On non monetary measures in the face of risks and the sign of the derivatives

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ON NON MONETARY MEASURES IN THE FACE OF RISKS AND THE SIGNS OF THE DERIVATIVES

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ABSTRACT

Vulnerability of both prudence and temperance towards a sure loss and towards a zero-mean background risk seems to be a very realistic assumption on individual’s preferences. This paper shows that when the concepts of prudence and temperance are defined in non-monetary terms, the above assumption is equivalent to the usual signs of the successive derivatives of the utility function.

Keywords: prudence, temperance, edginess, risk apportionment, utility premium
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I. INTRODUCTION

Within an expected utility framework, a common assumption is that the successive derivatives of the utility function $u$ alternate in signs\(^1\). The positive sign of the first derivative ($u' > 0$) captures the idea that a rational individual always prefers “more to less”. The concavity of $u$ ($u'' < 0$) is well known to define risk aversion (Pratt, 1964). Prudence and temperance are defined respectively by a positive third derivative ($u''' > 0$) and a negative fourth derivative ($u^{IV} < 0$) of the utility function. These assumptions are traditionally justified by reference to a specific decision problem: the analysis of precautionary savings for prudence (Kimball, 1990), and the demand for risky assets in the presence of background risks for temperance (Kimball, 1992). In the same vein, the concept of edginess ($u^V > 0$) was recently introduced by Chaherli-Lajeri (2004) to explain the effects of background risks on precautionary savings.

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\(^1\)This property is labelled “complete properness” by Pratt and Zeckhauser (1987) or “mixed risk aversion” by Caballé and Pomansky (1996).
Such explanation of the signs of the third, fourth and fifth derivatives of the utility function based upon specific decision models is in sharp contrast with the usual interpretation of the negative sign of the second derivative which relies on a very broad type of preference unrelated to a specific choice problem. Recently, Eeckhoudt and Schlesinger (2006) proposed a unified approach, based on preferences over specific classes of lotteries, to explain the meaning of prudence, temperance, edginess and higher derivatives. Their approach focused on the notion of risk apportionment. In line with this literature, the aim of this note is to propose an alternative interpretation of these signs that can be easily understood and remembered.

Our interpretation relies on two issues. On one hand, we extend two common hypotheses made on risk aversion to prudence and temperance. These two hypotheses, vulnerability of risk aversion towards a certain loss and towards a background risk, are known respectively as DARA (Decreasing Absolute Risk Aversion, Arrow (1970)) and risk vulnerability (Gollier and Pratt, 1996). As for risk aversion, it seems realistic to assume that prudence and temperance are both vulnerable to a sure loss and to a background risk. On the other hand, we formalize these two hypotheses using a non monetary measure of the concepts of prudence and temperance. As a matter of fact, in a seminal paper, Friedman and Savage (1948) introduced two measures to define risk aversion. The first one, the risk premium, later analysed by Pratt (1964), defines risk aversion as a monetary measure while the second, the utility premium, offers a measure of risk aversion in non monetary terms. Rebounding on the concept of the utility premium, a non monetary measure of prudence and temperance can be defined. We show that the necessary and sufficient conditions on the utility function for which these non monetary tools are vulnerable to a sure loss and to a background risk are equivalent to the usual signs of the fourth, fifth and sixth derivatives of the utility function expressed in the literature. In particular, we show that temperance is equivalent to prudence being vulnerable to a sure loss, and edginess is equivalent to prudence being vulnerable to a background risk.

This note is organised as follows. Section 2 presents a brief overview of the meaning of the signs of the successive derivatives of the utility function based on the notion of risk apportionment. Section 3 addresses the vulnerability of risk aversion to a sure loss and to a background risk. Section 4 and section 5 deal with the vulnerability of respectively prudence and temperance to a sure loss and to a background risk. Finally section 6 offers a short conclusion.

II. SIGNS OF THE DERIVATIVES AND RISK APPORTIONMENT

The positive sign of the third derivative has first been known as “downside risk aversion”, i.e. aversion to transferring a zero-mean risk from a richer to a poorer state (Menezes et al., 1980). Later, Kimball (1990) labelled \( u'''' > 0 \) as prudence and showed its equivalence with a precautionary saving motive, i.e. a decision maker is prudent when “an unavoidable risk leads him to increase his saving”\(^2\). Temperance, \( u^{IV} < 0 \), was first defined by Kimball (1992) in a context of risk management in the presence of background risk. A decision maker is temperant when “an unavoidable background risk leads him

\(^2\)Prudence has also been shown as describing preferences for some types of risk increase (Eeckhoudt et al., 1995).
to reduce exposure to another risk even if the two risks are statistically independent”\(^3\). Finally, edginess, defined by \(u^V > 0\), was addressed recently by Lajeri-Chaherli (2004) to explain the concept of standard prudence. A function exhibits standard prudence if “a consumption-decreasing risk and an independent risk has a precautionary interaction with losses is consumption decreasing”.

Such explanation of the signs of the third, fourth and fifth derivatives of the utility function based upon specific decision models is in sharp contrast with the usual interpretation of the negative sign of the second derivative which relies on a very broad type of preference unrelated to a specific choice problem. Eeckhoudt and Schlesinger (2006) proposed in a recent and original contribution a unified approach based on preferences over specific class of lotteries to explain the meaning of the signs of the successive derivatives of the utility function. It relies on the notion of risk apportionment or preferences for harm disaggregation. More precisely, the authors show that in an expected-utility framework with differentiable \(u\), risk apportionment of order \(n\) is equivalent to the condition:

\[
\text{sgn}(u^{(n)}) = \text{sgn}(-1)^{n+1}.
\]

Preference for harm disaggregation means that faced with two equally likely states of nature, a decision maker always prefers to receive one of two harms for certain (i.e. one in each state) as opposed to either facing the two harms jointly or facing none of them; the harm being either a sure loss or a zero-mean risk. In the case of risk apportionment of order 3, 4 and 5, Eeckhoudt and Schlesinger (2006) show, for lotteries with equiprobable lots, the following equivalences\(^4\):

\[
u'' > 0 \iff [-k, \tilde{\epsilon}] > [0, -k + \tilde{\epsilon}]
\]

\[
u^{IV} < 0 \iff [	ilde{\epsilon}_1, \tilde{\epsilon}_2] > [0, \tilde{\epsilon}_1 + \tilde{\epsilon}_2]
\]

\[
u^V > 0 \iff [0, -k + \tilde{\epsilon}_1, -k + \tilde{\epsilon}_2, \tilde{\epsilon}_1 + \tilde{\epsilon}_2] > [-k, \tilde{\epsilon}_1, \tilde{\epsilon}_2, -k + \tilde{\epsilon}_1 + \tilde{\epsilon}_2]
\]

While the passage from lotteries of equation (1) to lotteries of equation (2) can be easily understood, the passage from lotteries of equation (2) to lotteries of equation (3) is less evident. The purpose of this paper is to provide an alternative interpretation of the signs of the successive derivatives of the utility function that can be easily understood and remembered, without reference to any specific decision problem.

III. RISK AVERSION AND THE UTILITY PREMIUM

Under Expected Utility model, risk aversion is described by the concavity of \(u\) \((u'' < 0)\):

\[
u(x) > E[u(x + \tilde{\beta})]
\]

meaning that a risk averse individual prefers to receive zero instead of any zero-mean risk. Friedman and Savage (1948) introduce two ways for measuring risk aversion, or equivalently the cost of risk. The first way refers to a monetary measure, the risk premium, later analyzed by Pratt (1964) in a seminal paper. The risk premium \(\pi(x)\) is such that:

\[
E[u(x + \tilde{\beta})] = u(x - \pi(x))
\]

\(^3\)Eeckhoudt et al. (1995) showed that an individual is temperate if and only if marginal gains in expected utility for successive upwards shifts of any increase in risk are decreasing.

\(^4\)For higher orders, we refer the readers to Eeckhoudt and Schlesinger (2006).
\( \pi(x) \) is the amount of money that the agent is ready to pay to get rid of the risk \( \tilde{\beta} \). From Jensen inequality, \( \pi(x) > 0 \) if and only if \( u'' < 0 \). The second way refers to a non monetary measure of risk aversion, the utility premium, defined as:

\[
w(x) = u(x) - E[u(x + \tilde{\beta})]
\]  

(6)

\( w(x) \) measures the degree of “pain” associated with facing the risk \( \tilde{\beta} \), where pain is measured by the loss in expected utility from adding the risk \( \tilde{\beta} \) to wealth \( x \). From Jensen inequality, \( w(x) > 0 \) if and only if \( u'' < 0 \).

Two hypotheses on risk aversion have been dealt with in the literature. The first one is that risk aversion should decrease with wealth (Arrow, 1970) which is known as preferences being “decreasing absolute risk averse” (DARA). The second hypothesis is that risk aversion should increase with the introduction of an actuarial independent background risk, which is known as preferences being “risk vulnerable” (Gollier and Pratt, 1996). Note that the DARA assumption can also be described as the vulnerability of risk aversion towards a sure loss.

These two hypotheses have been addressed in the literature using the monetary measure of risk aversion, i.e. in terms of risk premium. In that case, they have been shown to be equivalent to very constraining necessary and sufficient conditions on the utility function. When addressed in terms of non monetary measure of risk aversion, necessary and sufficient conditions on the utility function are much simpler. In terms of utility premium, the vulnerability of risk aversion towards a sure loss, \( \delta > 0 \) (e.g. DARA hypothesis) writes as:

\[
w(x - \delta) > w(x)
\]  

(7)

Equation (7) is equivalent to \( w(x) \) decreasing. Using the definition of \( w(x) \), it is equivalent to \( u' \) convex i.e. to \( u'' > 0 \). Prudence is thus equivalent to risk aversion being vulnerable towards a sure loss. Note that the equivalence between (7) and the positive sign of \( u'' \) appeared in Eeckhoudt and Schlesinger (2006) in a different setting. It is interesting to stress that such equivalence corresponds exactly to the definition of prudence given by Kimball (1990).

In the same way, using the non monetary measure of risk aversion, the vulnerability of risk aversion towards a zero-mean background risk writes as:

\[
w(x + \tilde{\epsilon}) > w(x)
\]  

(8)

\( w(x + \tilde{\epsilon}) \) measures the degree of pain associated with facing the risk \( \tilde{\beta} \) in presence of the risk \( \tilde{\epsilon} \); its analytical expression is \( E\tilde{\epsilon}[u(x + \tilde{\epsilon})] - E\tilde{\epsilon}\tilde{\beta}[u(x + \tilde{\epsilon} + \tilde{\beta})] \), which equals \( E\epsilon[w(x + \tilde{\epsilon})] \) as \( \tilde{\beta} \) and \( \tilde{\epsilon} \) are independent. By Jensen’s inequality, equation (8) is thus equivalent to \( w \) convex, i.e. to \( u'' > 0 \) which writes as:

\[
u''(x) - E[u''(x + \tilde{\beta})] > 0
\]  

(9)

See also Stone (1970), Menegatti (2007) and Eeckhoudt, Rey and Schlesinger (2007) for related papers using the utility premium concept.

6The general definition considers an unfair background risk, i.e. a risk with a non positive mean.

7In that case, DARA writes as \( \pi(x - \delta) > \pi(x) \) with \( \delta > 0 \), and risk vulnerability writes as \( \pi(x + \tilde{\epsilon}) > \pi(x) \) with \( \tilde{\epsilon} \) and \( \tilde{\beta} \) independent.

8Indeed, equation (7) rewrites as \( w'(x) < 0 \), or equivalently as \( u'(x) = u'(x + \tilde{\beta}) < 0 \) which is the characterization of prudence given by Kimball (1990) in terms of precautionary savings.
Using once again Jensen’s inequality, equation (9) is equivalent to \( u'' \) concave, i.e. to \( u^{IV} < 0 \). The equivalence between the convexity of the utility premium and the negative sign of \( u^{IV} \) appeared in Eeckhoudt and Schlesinger (2006). We provide here another interpretation of the convexity of the utility premium in the sense that temperance is equivalent to risk aversion being vulnerable to a zero-mean background risk. Let us stress that equation (9) corresponds exactly to the definition of temperance given by Kimball (1992).

Rebounding on the concept of utility premium, the next sections address non monetary measures of prudence and temperance. They provide conditions on the utility function under which these measures are vulnerable towards a sure loss and towards a background risk.

IV. PRUDENCE UTILITY PREMIUM

Prudence is known as preference for a zero mean risk in the wealthier state of nature. This writes as:

\[
u(x - k) + E[u(x + \beta)] > u(x) + E[u(x - k + \beta)]
\]

(Crainich and Eeckhoudt (2008) use the characterization of prudence given by equation (10) to propose a new monetary measure of prudence. The idea here is to use such characterization to propose a non monetary measure of prudence. In the same way as Friedman and Savage (1948) defined the utility premium as a non monetary measure of risk aversion, we propose the **prudence utility premium**, denoted \( w_P(x) \), to define a non monetary measure of prudence:

\[
w_P(x) = u(x - k) + E[u(x + \beta)] - u(x) - E[u(x - k + \beta)]
\]

\( w_P(x) \) measures the degree of pain of facing the risk in the poorer state compared to facing it in the wealthier state, or following Eeckhoudt and Schlesinger (2006) terminology, the degree of pain due to misapportionment of order 3. Naturally, \( w_P(x) > 0 \) if and only if \( u'' > 0 \).

We will now show that assuming vulnerability of prudence towards a sure loss and towards a background risk is equivalent to having the utility function verifying respectively temperance and edginess.

When addressed in non monetary terms, having prudence vulnerable to a sure loss writes as:

\[
w_P(x - \delta) > w_P(x)
\]

To show that equation (12) is equivalent to \( u^{IV} < 0 \), we use the fact that \( w_P(x) = w(x - k) - w(x) \) to rewrite equation (12) as:

\[
w(x - k - \delta) - w(x - \delta) > w(x - k) - w(x)
\]

Equation (13) means that \( w \) is convex which is equivalent to \( u'' \) concave i.e. to \( u^{IV} < 0 \).

In the same way, prudence is vulnerable to any zero-mean background risk if and only if:

\[
w_P(x + \bar{\epsilon}) > w_P(x)
\]
To show that equation (14) is equivalent to $u^V > 0$, we use the fact that $w_P(x) = w(x - k) - w(x)$ to rewrite equation (14) as:

$$E[w(x - k + \tilde{\epsilon})] - E[w(x + \tilde{\epsilon})] > w(x - k) - w(x) \tag{15}$$

Let’s denote $g(x) = w(x) - E[w(x + \tilde{\epsilon})]$, equation (15) rewrites as:

$$g(x) > g(x - k) \tag{16}$$

which is equivalent to $g' > 0$, i.e. to:

$$w'(x) > E[w'(x + \tilde{\epsilon})] \tag{17}$$

From Jensen’s inequality, equation (17) means that $w'$ is concave which writes as:

$$u'''(x) < E[u'''(x + \tilde{\beta})], \tag{18}$$

Using once again Jensen’s inequality, equation (18) is equivalent to $u'''$ convex, i.e. to $u^V > 0$.

These results provide a new and intuitive characterization of temperance and edginess.

**Proposition 1.**

Temperance $(u^IV < 0)$ is equivalent to prudence being vulnerable to a sure loss.

Edginess $(u^V > 0)$ is equivalent to prudence being vulnerable to any zero-mean background risk.

## V. TEMPERANCE UTILITY PREMIUM

As for prudence, we can define a non monetary measure of temperance. Temperance is known as a preference for disaggregation of two independent zero-mean risks:

$$E[u(x + \tilde{\theta})] + E[u(x + \tilde{\beta})] > u(x) + E[u(x + \tilde{\theta} + \tilde{\beta})] \tag{19}$$

We define the temperance utility premium, denoted $w_T(x)$ as follows:

$$w_T(x) = E[u(x + \tilde{\theta})] + E[u(x + \tilde{\beta})] - u(x) - E[u(x + \tilde{\theta} + \tilde{\beta})] \tag{20}$$

$w_T(x)$ measures the degree of pain of facing a risk in the presence of another, or following Eeckhoudt and Schlesinger (2006) terminology, the degree of pain due to a misapportionment of order 4. Naturally, $w_T(x) > 0$ if and only if $u^IV < 0$.

We will now show that assuming vulnerability of temperance towards a sure loss and towards a background risk is equivalent to having the utility function verifying respectively $u^V > 0$ and $u^VI < 0$.

The vulnerability of temperance towards a sure loss writes as:

$$w_T(x - \delta) > w_T(x) \tag{21}$$
To show that equation (21) is equivalent to $u^V > 0$, we use the fact that $w_T(x) = E[w(x + \tilde{\theta})] - w(x)$ to rewrite equation (21) as:

$$E[w(x + \tilde{\theta} - \delta)] - w(x - \delta) > E[w(x + \bar{\theta})] - w(x)$$  \hspace{1cm} (22)

Replacing $\delta$ by $k$, and replacing $\tilde{\theta}$ by $\hat{\theta}$, equation (21) is equivalent to equation (15) that was previously shown to be equivalent to $u^V > 0$.

In the same vein, temperance is vulnerable to any zero-mean background risk if and only if:

$$w_T(x + \hat{\varepsilon}) > w_T(x)$$  \hspace{1cm} (23)

To show that equation (23) is equivalent to $u^V I < 0$, we use the fact that $w_T(x) = E[w(x + \hat{\theta})] - w(x)$ to rewrite equation (23) as:

$$w(x) - E[w(x + \hat{\varepsilon})] > E[w(x + \hat{\theta})] - E[w(x + \hat{\theta} + \hat{\varepsilon})]$$  \hspace{1cm} (24)

Using the expression of $g$ defined in the previous section, equation (24) rewrites as:

$$g(x) > E[g(x + \hat{\theta})]$$  \hspace{1cm} (25)

which is equivalent to $g'' < 0$, i.e. to:

$$w''(x) < E[w''(x + \hat{\varepsilon})]$$  \hspace{1cm} (26)

By Jensen’s inequality, this last equation means that $w''$ is convex which writes as:

$$u^{IV}(x) > E[u^{IV}(x + \hat{\beta})]$$  \hspace{1cm} (27)

which is equivalent to $u^{IV}$ concave, i.e. to $u^V I < 0$. We obtain the following result.

**Proposition 2.**

Edginess ($u^V > 0$) is equivalent to temperance being vulnerable to a sure loss.

$u^V I < 0$ is equivalent to temperance being vulnerable to any zero-mean background risk.

Before concluding, let us make two remarks.

First, by iteration, we could define a measure of pain due to misapportionment of order $n$ and look at the vulnerability of this measure towards a sure loss and a background risk. This would make it possible to generalise our previous propositions and would give the equivalence between the three following items ($\forall n \geq 3$):

(i) risk apportionment of order $n$ holds,
(ii) risk apportionment of order $(n - 1)$ is vulnerable to a sure loss, and
(iii) risk apportionment of order $(n - 2)$ is vulnerable to a zero-mean background risk.

Second, it is important to stress that the conditions given in propositions 1 and 2 are both necessary and sufficient. If prudence and temperance were defined in monetary terms, these conditions would only be necessary. Consider risk aversion as an illustration. It is well known that $u''' > 0$ and $u^{IV} < 0$ are only necessary conditions to have, respectively, DARA and risk vulnerability when risk aversion is measured in monetary terms.

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9Risk apportionment of order 1 is equivalent to $u' > 0$. We can define a measure of pain of facing a loss $k > 0$ as $w_R(x) = u(x) - u(x - k)$. Naturally $w_R(x) > 0$ if and only if $u' > 0$.  

7
while these two conditions are both necessary and sufficient when risk aversion is defined in non monetary terms, as shown in section 3. By iteration, the same applies to prudence and temperance.

VI. CONCLUSION

Vulnerability of risk aversion towards a sure loss and towards a zero-mean background risk are two common hypotheses in the literature. In the same vein, vulnerability of both prudence and temperance towards a sure loss and towards a zero-mean background risk seems to be a very realistic assumption on individual’s preferences. This paper has shown that when the concepts of prudence and temperance are defined in non monetary terms, the above assumptions are equivalent to the usual signs of the successive derivatives of the utility function. These results offer an alternative interpretation of the signs of the successive derivatives which can be easily understood and remembered, without reference to any specific decision problem.

REFERENCES


