Prudence, temperance, edginess and higher degree risk apportionment as decreasing correlation aversion

- Michel DENUIT (Université Catholique Louvain, Belgique)
- Béatrice REY (Université Lyon 1, Laboratoire SAF)

2009.10 (WP 2110)
PRUDENCE, TEMPERANCE, EDGINESS, AND RISK APPORTIONMENT AS DECREASING SENSITIVITY TO DETRIMENTAL CHANGES

MICHÉL DENUIT
Institut de Sciences Actuarielles & Institut de Statistique
Université Catholique de Louvain, Louvain-la-Neuve, Belgium
michel.denuit@uclouvain.be

BÉATRICE REY
Institut de Science Financière et d’Assurances
Université de Lyon, Université Lyon 1, Lyon, France
rey-fournier@univ-lyon1.fr

November 2009
Abstract

This paper shows that the notions of prudence, temperance, edginess, and, more generally, risk apportionment of any degree are the consequences of the natural idea that the sensitivity to detrimental changes should decrease with initial wealth. In the setting of Epstein & Tanny (1980), this turns out to be equivalent to the supermodularity of the expected utility for some specific 4-state lotteries.

JEL classification: D81

Key words: expected utility, wealth effect, supermodularity, stochastic dominance.
1 Introduction and motivation

TEXT TO BE ADDED...

In this paper we show that the natural feeling of a sensitivity to detrimental changes decreasing with initial wealth can be used to explain the notions of prudence, temperance, and edginess which are now often used in the analysis of risky choices besides that of risk aversion. Formally in the expected utility model prudence, temperance, and edginess are defined respectively by a positive third derivative, by a negative fourth derivative, and by a positive fifth derivative of the utility function. Note that these concepts appear at least indirectly in non-expected utility models (Bleichrodt & Eeckhoudt (2005)). These assumptions are traditionally justified by reference to a specific decision problem: the analysis of precautionary savings for prudence¹ (Kimball (1990)), the demand for risky assets in the presence of background risks for temperance (Kimball (1992), Gollier & Pratt (1996)), and the reactivity to multiple risks on precautionary motives for edginess (Lajeri-Chaherli (2004)). This explanation of the sign of the third, the fourth, and the fifth derivatives of the utility function based upon specific decision models is in sharp contrast with the usual interpretation of the negative sign of the second derivative which relies on a very broad type of preference unrelated to a specific choice problem. In this paper, we show that risk apportionment of any degree can be interpreted as a lower sensitivity to detrimental changes when the decision-maker gets richer. Then, using the elementary correlation increasing transformation defined by Epstein & Tanny (1980) we show that prudence, temperance, and edginess are based on the natural idea that aversion to probability spreads in specific 4-state lotteries should decrease as wealth increases. From a mathematical point of view, this amounts to require that the expected utility is supermodular in the initial wealth level and Epstein-Tanny correlation parameter when the decision-maker is faced with these specific lotteries.

Starting from a different premise, Eeckhoudt & Schlesinger (2006) and Eeckhoudt et al. (2009) also justify prudence and temperance on the basis of another general

¹The role of prudence has also been recently illustrated in other contexts: self-protection activities (Chiu (2005)), optimal audits (Fagart & Sinclair-Desgagné (2007))
preference. In the first paper they state it as a preference for “pain disaggregation” while in the second one they rely upon the tendency to “combine good with bad”. Notice that these two papers include references to previous papers that had partially used similar ideas. The idea of decreasing sensitivity to detrimental changes developed here encompasses the other ones. In this paper, we show that correlation aversion as well as risk apportionment both result from the natural tendency of getting less sensitive to to detrimental changes as wealth increases. The signs of the successive derivatives ensure that the dislike for aggregating harms as well as the attractiveness of combining good with bad are both decreasing with wealth.

The present work is organized as follows. In Section 2, we first introduce some notations needed in the paper. Then, we prove that the sign of the successive derivatives of the utility function control the monotonicity of the aversion to detrimental changes. In Section 3, we present the concept of an “elementary correlation increasing transformation” and we recall the seminal result by Epstein & Tanny (1980) relating risk aversion to “(positive) correlation aversion”. Measuring the dislike for correlation by the approach based on utility premium developed after Friedman & Savage (1948) (see also Eeckhoudt & Schlesinger (2006)), we show that prudence, like risk aversion, is a consequence of the intuitive idea that a decision-maker should be less sensitive to an increase in correlation when he gets richer. This result is then extended to general risk apportionments in Section 3. Given the importance of the concept of temperance and edginess, their equivalence to decreasing correlation aversion is discussed in details in Section 4. The closing Section 5 briefly concludes the paper.

2 Decreasing sensitivity to detrimental changes

2.1 Notation

Henceforth, we denote as \( u', u'', u''' \) the first derivative, the second derivative, and the third derivative of the utility function \( u \). More generally, we write \( u^{(n)} \) for the \( n \)th derivative of \( u \), \( n = 1, 2, 3, 4, \ldots \); the notations \( u', u'', u''' \) and \( u^{(1)}, u^{(2)}, u^{(3)} \), respectively, will
be used interchangeably. As decision-makers are usually assumed to be non-satiated and risk-averse, $u$ is non-decreasing and concave. If $u$ is differentiable, this means that $u' \geq 0$ and $u'' \leq 0$.

More recently, it has been shown that higher order derivatives of $u$ also matter. Therefore, let us consider the non-decreasing utility functions with derivatives of degrees 1 to $s$ of alternating signs. This property is satisfied by the utility functions most commonly used in mathematical economics including all the completely monotone utility functions such as the logarithmic, exponential and power utility functions. Formally, let us define the class $U_{s-icv}$, $s = 1, 2, \ldots$, of the regular $s$-increasing concave function as the class containing all the utility functions such that $(-1)^k u^{(k)} \geq 0$ for $k = 1, \ldots, s$. Elements of $U_{s-icv}$ are said to be $s$-increasing concave. To get all the $s$-increasing concave utilities, we need to supplement $U_{s-icv}$ with all the pointwise limits of elements in $U_{s-icv}$. This gives the class $\overline{U}_{s-icv}$ of all the utilities such that $(-1)^k u^{(k)} \geq 0$ for $k = 1, \ldots, s - 2$ and $(-1)^{s-1} u^{(s-2)}$ is non-decreasing and concave.

The class $\overline{U}_{s-icv}$ can be characterized by sign properties of divided differences. Recall that the $k$th divided difference, $k \geq 1$, of the function $u$ at distinct points $x_0, x_1, \ldots, x_k$, denoted by $[x_0, x_1, \ldots, x_k]u$, is defined recursively by

$$[x_0, x_1, \ldots, x_k]u = \frac{[x_1, x_2, \ldots, x_k]u - [x_0, x_1, \ldots, x_{k-1}]u}{x_k - x_0}, \quad (2.1)$$

starting from $[x_i]u = u(x_i)$, $i = 0, 1, \ldots, k$. These divided differences extend derivatives to less regular functions. Then, $u \in \overline{U}_{s-icv}$ if, and only if, $(-1)^{k+1}[x_0, x_1, \ldots, x_k]u \geq 0$ for any $x_1, x_1, \ldots, x_k$, $k = 1, 2, \ldots, s$.

The class $\overline{U}_{s-icv}$ of the $s$-increasing concave functions is the largest class of functions for which the implication $X \preceq_{s-icv} Y \Rightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ holds true for every pair $(X, Y)$ of ordered random variables. For this reason, $\overline{U}_{s-icv}$ is often called the maximal generator of the order $\preceq_{s-icv}$. This means that $\overline{U}_{s-icv}$ corresponds to the largest class of decision-makers whose preferences are in accordance with $\preceq_{s-icv}$. We refer the reader, e.g., to DENUIT, DE VIJLDER & LEFÈVRE (1999) for more details about $\preceq_{s-icv}$.

Letting $s$ tend to $+\infty$ gives utilities with all odd derivatives positive and all even deriva-
tives negative. In this case, utility functions are completely monotone and express mixed risk aversion, as studied in Caballé & Pomansky (1996).

2.2 Higher degree stochastic dominance relations

The common preferences of all the decision-makers with \( s \)-increasing concave utility functions generate the \( s \)-increasing concave dominance rule, called the \( s \)-increasing concave order. More precisely, given two random variables \( X \) and \( Y \), \( X \) is said to be smaller than \( Y \) in the \( s \)-increasing concave order, denoted by \( X \preceq_{s-icv} Y \) when

\[
\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-icv}
\]

\( \iff \)

\[
\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-icv},
\]

provided the expectations exist. For more details about these orders, we refer the interested readers to Denuit, Lefèvre & Shaked (1998) and Denuit, De Vijlder & Lefèvre (1999).

These orders are closely related to the \( st \)th degree increase in risk of Ekern (1980), denoted here as \( \preceq_{s-cv} \). Specifically,

\[
X \preceq_{s-icv} Y \quad \mathbb{E}[X^k] = \mathbb{E}[Y^k] \quad \iff \quad \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ such that } (-1)^{s+1}u^{(s)}(s) \geq 0.
\]

If we define as \( \mathcal{U}_{s-cv} \) the class of the regular \( s \)-concave utilities, i.e. those with \( (-1)^{s+1}u^{(s)}(s) \geq 0 \), and as \( \overline{\mathcal{U}}_{s-cv} \) the class of all the \( s \)-concave utilities, i.e. those such that \( (-1)^{s-1}u^{(s-2)} \) is concave we can then define the \( s \)-concave orders \( \preceq_{s-cv} \) as

\[
X \preceq_{s-cv} Y \quad \iff \quad \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \overline{\mathcal{U}}_{s-cv}
\]

\( \iff \)

\[
\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \text{ for all } u \text{ in } \mathcal{U}_{s-cv}
\]

\( \iff \)

\[
X \preceq_{s-icv} Y \text{ and } \mathbb{E}[X^k] = \mathbb{E}[Y^k] \text{ for } k = 1, 2, \ldots, s - 1
\]

2.3 Aversion to detrimental changes and \( s \)-increasing utility functions

The following result is at the core of our analysis. It states that a decision-maker with a \( s \)-increasing concave utility function becomes less sensitive to detrimental changes as wealth
increases. Using this result, we will be able to examine how the preference for risk apportionment is changing with wealth.

We know from the proof of Theorem 3 in Eeckhoudt, Schlesinger & Tsetlin (2009) that given \( u \in U(s+1)_{icv} \) and \( X \preceq_s Y \) the function \( g \) defined by

\[
g(w) = E[\{u(w + X)\}] - E[\{u(w + Y)\}]
\]

belongs to \( U_{s-icv} \). In the next result, we study the equivalence between the non-decreasingness of \( g \) and \( u \in U(s)_{icv} \), thus allowing for utilities with non-differentiable \( u(s-2) \).

**Proposition 2.1.** Consider \( X \preceq_s Y, u \in U(s)_{icv}, \) and \( g \) defined in (2.2). Then, \( u \in U(s)_{icv} \Rightarrow g \) non-decreasing. Conversely, if whatever \( X \) and \( Y \) such that \( X \preceq_s Y \), \( g \) defined in (2.2) is non-decreasing then \( u \in U(s)_{icv} \).

**Proof.** Note that \( u \in U(s+1)_{icv} \subset U(s)_{icv} \Rightarrow g \leq 0 \). Define for \( h \geq 0 \) the function \( \Delta_h u \) as

\[
\Delta_h u(w) = u(w + h) - u(w).
\]

Recall that if \( u \in U(2j+1)_{icv} \) then \([x_0, \ldots, x_{2j}]u\) is non-decreasing in \( x_0, \ldots, x_{2j} \) whereas if \( u \in U(2j)_{icv} \) then \([x_0, \ldots, x_{2j-1}]u\) is non-decreasing in \( x_0, \ldots, x_{2j-1} \). Hence, \([x_0, \ldots, x_k](\Delta_h u)\) is non-negative if \( k = 2j-1 \) and non-positive if \( k = 2j \) for \( k = 1, \ldots, s \), that is, \( \Delta_h u \in U(s)_{icv} \). Hence,

\[
X \preceq_s Y \Leftrightarrow E[-\Delta_h u(w + X)] \leq E[-\Delta_h u(w + Y)]
\]

\[
\Leftrightarrow E[u(w + X)] - E[u(w + h + X)] \leq E[u(w + Y)] - E[u(w + h + Y)]
\]

\[
\Leftrightarrow g(w) \leq g(w + h)
\]

so that \( g \) is non-decreasing.

Now, \( g(w + h) \geq g(w) \) for any \( h \geq 0 \) means that

\[
E[-\Delta_h u(w + X)] \leq E[-\Delta_h u(w + Y)]
\]

holds for any \( h \geq 0 \) and for any ordered pair \((X, Y)\). Let us now prove that \( -\Delta_h u \in U(s)_{icv} \). To this end, we show that if \( -\Delta_h u \) does not belong to \( U(s)_{icv} \) then we can two two
random variables $X$ and $Y$ such that $X \preceq_{s-icv} Y$ but $\mathbb{E}[-\Delta_h u(w + X)] > \mathbb{E}[-\Delta_h u(w + Y)]$. If $-\Delta_h u \not\in \mathbb{U}_{s-icv}$, one can find $(s + 1)$ distinct points $x_0 < x_1 < \ldots < x_s$ such that $(-1)^{s+1}[x_0, x_1, \ldots, x_s](-\Delta_h u) < 0$. Expanding the divided difference, this means that

$$(-1)^{s+1} \sum_{j=0}^{s} \frac{-\Delta_h u(x_j)}{(x_j - x_0) \ldots (x_j - x_{j-1})(x_j - x_{j+1}) \ldots (x_j - x_s)} < 0. \quad (2.3)$$

Now, let $\mu$ be the measure defined by

$$\mu(\{x_j\}) = \frac{(-1)^{s+1}}{(x_j - x_0) \ldots (x_j - x_{j-1})(x_j - x_{j+1}) \ldots (x_j - x_s)} \text{ for } j = 0, 1, \ldots, s, \quad (2.4)$$

and which is given for any subset $A$ of $[a, b]$ by $\mu(A) = \sum_{x_j \in A} \mu(\{x_j\})$. Denote as $\psi_j$ the function $x \mapsto \psi_j(x) = x^j$ with the understanding that $\psi_0$ is constantly equal to 1. Now,

$$\sum_{j=0}^{s} (-1)^{s+1} \frac{(x_j - x_0) \ldots (x_j - x_{j-1})(x_j - x_{j+1}) \ldots (x_j - x_s)}{(x_j - x_0) \ldots (x_j - x_{j-1})(x_j - x_{j+1}) \ldots (x_j - x_s)} = (-1)^{s+1}[x_0, x_1, \ldots, x_s] \psi_0 = 0$$

so that $\mu$ is a signed measure on $[a, b]$ such that $\mu([a, b]) = 0$ and $\mu(\{x_j\}) \neq 0$ for $j = 0, 1, \ldots, s$. Hence, $\mu$ can be represented as the difference $\mu_1 - \mu_2$ between two non-negative measures $\mu_1$ and $\mu_2$ with an equal total mass. We can thus define two sequences of non-negative real numbers $p_j$ and $q_j$, $j = 0, 1, \ldots, s$, each of sum 1 and such that $p_j - q_j = \alpha \mu(\{x_j\})$ for $j = 0, 1, \ldots, s$, $\alpha$ being some positive normalizing constant. Now, let us introduce two discrete random variables $X$ and $Y$ with the same support $\{x_0, x_1, \ldots, x_s\}$ and with masses $\operatorname{Pr}[X = x_j] = q_j$ and $\operatorname{Pr}[Y = x_j] = p_j$, $j = 0, 1, \ldots, s$. We first show that $X \preceq_{s-icv} Y$ holds. By construction, we have

$$\mathbb{E}[Y^k] - \mathbb{E}[X^k] = (-1)^{s+1} \alpha \sum_{j=0}^{s} \frac{x_j^k}{(x_j - x_0) \ldots (x_j - x_{j-1})(x_j - x_{j+1}) \ldots (x_j - x_s)}$$

$$= (-1)^{s+1} \alpha [x_0, x_1, \ldots, x_s] \psi_k = 0$$

for $k = 0, 1, \ldots, s - 1$, so that $F_X - F_Y$ has $s - 1$ sign changes. Furthermore, we see that

$$p_s - q_s = \frac{(-1)^{s+1} \alpha}{(x_s - x_0)(x_s - x_1) \ldots (x_s - x_{s-1})}$$

which implies that $F_X \geq F_Y$ on some right-interval $[\eta, b]$ for $s$ odd and $F_X \leq F_Y$ on some right-interval $[\eta, b]$ for $s$ even. By the crossing condition, this implies that $X \preceq_{s-cv} Y$ so that $X \preceq_{s-icv} Y$. Then, it remains to note that again by construction,

$$\mathbb{E}[-\Delta_h u(Y)] - \mathbb{E}[-\Delta_h u(X)] = (-1)^{s+1} \alpha [x_0, x_1, \ldots, x_s] g < 0. \quad (2.6)$$
Thus, $-\Delta_h u \in \mathcal{U}_{s-icv}$ for any $h \geq 0$. To prove that $u \in \mathcal{U}_{(s+1)-icv}$, we need to establish that $(-1)^s u^{(s-1)}$ is non-decreasing and concave, or equivalently that the increments

$$(-1)^s \left( u^{(s-1)}(w + h) - u^{(s-1)}(w) \right)$$

of $(-1)^s u^{(s-1)}$ are non-negative and non-increasing. This is indeed the case since $-\Delta_h u \in \mathcal{U}_{s-icv}$ implies that

$$w \mapsto (-1)^{s-1}(-\Delta_h u)^{(s-2)}(w) = (-1)^s \left( u^{(s-2)}(w + h) - u^{(s-2)}(w) \right)$$

is non-decreasing and concave.

We see that the pain $E[u(w + X)] - E[u(w + Y)]$ caused by the deterioration of $Y$ into $X$ decreases as the initial wealth $w$ increases. The decision-maker thus becomes less sensitive to detrimental changes of $Y$ into $X$ as he gets richer.

3 Correlation aversion, risk aversion and prudence

3.1 Elementary correlation increasing transformation

Let us recall the concept of an “elementary correlation increasing transformation”. This concept links correlation aversion to risk aversion, as was already shown in Epstein & Tanny (1980, Theorem 4), and is at the core of the present paper. Let $I_1$ and $I_2$ be a couple of binary random variables such that

$$\Pr[I_i = 0] = 1 - \Pr[I_i = 1] = p_i, \ i = 1, 2.$$ 

Without loss of generality, we assume that $p_1 \leq p_2$. Let us now consider $\rho$ such that $-p_1 p_2 \leq \rho \leq p_1(1 - p_2)$ and define the joint distribution of $(I_1, I_2)$ as

$$\begin{align*}
\Pr[I_1 = 0, I_2 = 0] &= p_1 p_2 + \rho \\
\Pr[I_1 = 1, I_2 = 0] &= (1 - p_1) p_2 - \rho \\
\Pr[I_1 = 0, I_2 = 1] &= p_1 (1 - p_2) - \rho \\
\Pr[I_1 = 1, I_2 = 1] &= (1 - p_1)(1 - p_2) + \rho.
\end{align*}$$
Compared to the case when $I_1$ and $I_2$ are mutually independent, we see that $\rho$ is added to the probability mass at $(0,0)$ and $(1,1)$, whereas the same quantity is subtracted from the probability mass at $(0,1)$ and $(1,0)$. Clearly,

$$\text{Cov}[I_1, I_2] = \text{Pr}[I_1 = 1, I_2 = 1] - \text{Pr}[I_1 = 1] \text{Pr}[I_2 = 1] = \rho$$

so that $\rho$ can be considered as a correlation parameter.

When $\rho$ increases we face a correlation increasing transformation as defined by Epstein & Tanny (1980) and a correlation averse decision-maker should then dislike an increase in $\rho$. Let us now prove that correlation aversion implies risk aversion in the expected utility model. Consider a decision-maker with utility function $u$ and initial wealth $w$ facing the risky outcome $a_1I_1 + a_2I_2$ for some non-negative constants $a_1$ and $a_2$. Clearly, $a_1I_1 + a_2I_2$ corresponds to the 4-state lottery

$$a_1I_1 + a_2I_2 = \begin{cases} 
0 & \text{with probability } p_1p_2 + \rho \\
a_1 & \text{with probability } (1 - p_1)p_2 - \rho \\
a_2 & \text{with probability } p_1(1 - p_2) - \rho \\
a_1 + a_2 & \text{with probability } (1 - p_1)(1 - p_2) + \rho.
\end{cases}$$

The corresponding expected utility is

$$U(w, \rho) = \mathbb{E}[u(w + a_1I_1 + a_2I_2)]$$

$$= (p_1p_2 + \rho)u(w) + ((1 - p_1)p_2 - \rho)u(w + a_1) + (p_1(1 - p_2) - \rho)u(w + a_2) + ((1 - p_1)(1 - p_2) + \rho)u(w + a_1 + a_2).$$

(3.1)

It is easily seen that $U(w, \rho)$ is non-increasing in $\rho$ when $u$ is concave. Indeed the partial derivative of $U(w, \rho)$ with respect to $\rho$ equals

$$\frac{\partial}{\partial \rho} U(w, \rho) = u(w + a_1 + a_2) - u(w + a_1) - (u(w + a_2) - u(w))$$

(3.2)

which is non-positive when $u$ is concave (so that marginal utility is non-increasing). This shows that an increase in the correlation parameter $\rho$ is welfare deteriorating for a risk-averse decision-maker, as pointed out by Epstein & Tanny (1980).

---

2A similar set-up is used by Doherty & Schlesinger (1983) but their objective was quite different from ours.
As explained in the introduction, we measure here the strength of dislike for correlation by means of a correlation utility premium defined as

\[ CUP(w, \rho) = U(w, \rho) - U(w, 0). \]

In words, \( CUP(w, \rho) \) measures the degree of “pain” associated with facing the correlation \( \rho \), where pain is measured by the loss in expected utility resulting from the correlation \( \rho \) between the random variables \( a_1 I_1 \) and \( a_2 I_2 \) compared to independence. Considering (3.1), we see that

\[ CUP(w, \rho) = \rho \left( u(w + a_1 + a_2) - u(w + a_1) - u(w + a_2) + u(w) \right) \]

so that \( \frac{\partial CUP(w, \rho)}{\partial \rho} \leq 0 \) for all \( w, a_1, a_2 \) if, and only if, \( u'' \leq 0 \).

As pointed out by Friedman & Savage (1948) for the cost of risk, there are to ways for measuring the impact of the correlation. The first way refers to a monetary measure, the correlation premium \( \pi(w, \rho) \) such that \( U(w, \rho) = U(w - \pi(w, \rho), 0) \). Here, \( \pi(w, \rho) \) is the amount of money that the agent is ready to pay to eliminate the correlation level between risks. The second way refers to a non monetary measure, the “correlation utility premium” \( CUP(w, \rho) \) defined above \( \hat{CUP}(w, \rho) = U(w, 0) - U(w, \rho) \). It measures the degree of “pain” e.g. the disutility associated with facing the correlation \( \rho \). Note that

\[ \text{sign}(CUP(w, \rho)) = \text{sign}(\pi(w, \rho)). \]

We refer the reader, e.g., to Jindapon & Neilson (2007) for an extensive discussion about these two ways of measuring the cost of a deterioration in the decision-maker’s wealth.

To propose an interpretation more grounded on observable data for the sensitivity to an increase in correlation, let us consider the willingness to pay to decrease the correlation. Should \( \rho \) be transformed into \( \rho_0 \) with \( \rho_0 < \rho \), expected utility would remain constant if the wealth level \( w \) were changed by a compensating variation \( v \) such that:

\[ U(w, \rho) = U(w - v, \rho_0). \]
When $\rho_0$ is marginally changed around $\rho$, the willingness to pay is given by a total differentiation of equation (3.1), that is,

$$ WTP_\rho = \frac{dW}{d\rho} = -\frac{\partial U(w, \rho)}{\partial \rho} \cdot \frac{\partial U(w, \rho)}{\partial w}. $$

Thus, $WTP_\rho$ is defined by the marginal rate of substitution between wealth $w$ and the correlation level $\rho$. It captures the tradeoff between a change in wealth and a change in correlation level. Notice that the sign of $WTP_\rho$ is the sign of $-\frac{\partial U(w, \rho)}{\partial \rho}$ that coincides with the sign of $-\frac{\partial U(w, \rho)}{\partial \rho}$.

### 3.2 Prudence

As indicated in the introduction prudence is defined by the non-negativity of the third derivative of the utility function. It is usually justified by reference to the decision of building up precautionary savings in order to better face future income risk. We now show that in fact prudence, like risk aversion, can be justified by the decision-maker’s attitude to an increase in the correlation parameter $\rho$.

In order to stress the intuitive nature of the concept of prudence, let us notice that it is pretty reasonable to assume that a decision-maker becomes less sensitive to an increase in the correlation parameter when he is richer, i.e. an increase in the initial wealth $w$ should moderate the negative impact of an higher value of $\rho$ on welfare.

To analyze the implications of this assumption, let us consider the random variable $I_1 a_1 + I_2 a_2$ where $I_1$ and $I_2$ are as described in Section 2.2. For given positive values of $a_1$ and $a_2$, an increase in $\rho$ should reduce welfare less when $w$ is large since then it affects a smaller share of the initial wealth. Considering that under correlation aversion the derivative of the expected utility $U(w, \rho)$ is negative and that this derivative should approach 0 as $w$ increases, this means that we expect

$$ \frac{\partial^2}{\partial w \partial \rho} U(w, \rho) = \frac{\partial}{\partial w} \left( \frac{\partial}{\partial \rho} E[u(w + a_1 I_1 + a_2 I_2)] \right) \geq 0, \quad (3.3) $$

that is, the function $(w, \rho) \mapsto U(w, \rho)$ is supermodular. This derivative equals

$$ \frac{\partial^2}{\partial w \partial \rho} U(w, \rho) = u'(w) + u'(w + a_1 + a_2) - u'(w + a_1) - u'(w + a_2), \quad (3.4) $$

10
which is non-negative when \( u' \) is convex \( \iff u \in U_{3-icv} \). If \( u \) is thrice differentiable, \( u' \) is convex \( \iff u'' \geq 0 \). Consequently, prudence can also be interpreted as an implication of the lower sensitivity to an increase in \( \rho \) due to increased initial wealth.

Coming back to the correlation utility premium, we see that the sign of \( \frac{\partial^2}{\partial w \partial \rho} CUP(w, \rho) \) coincides with the sign of \( \frac{\partial^2}{\partial w \partial \rho} U(w, \rho) \) and they both relate to the sign of \( u''' \). Note that this sign also indicates how the willingness to pay varies with the correlation level \( \rho \) as for a risk averse agent

\[
\text{sign} \left( \frac{\partial WTP}{\partial \rho} \right) = \text{sign} \left( \left( \frac{\partial U(w, \rho)}{\partial \rho} \right) \left( \frac{\partial^2 U(w, \rho)}{\partial w \partial \rho} \right) \right) .
\] (3.5)

4 Risk apportionment of higher degrees

4.1 Decreasing aversion to probability spreads in 4-state lotteries

Considering Section 2, we know that risk aversion means that the decision-maker dislikes an increase in the correlation parameter \( \rho \) when final wealth is given by \( w + a_1 I_1 + a_2 I_2 \). Faced with the same final wealth, prudence means that the decision-maker is less sensitive to an increase in \( \rho \) when he gets richer. This section shows that the same idea can be used to characterize temperance, edginess, and higher degree risk apportionment, substituting more general random variables for \( w + a_1 I_1 + a_2 I_2 \). Specifically, we show that any risk apportionment can be defined as a lower sensitivity to an increase in the correlation parameter \( \rho \) as wealth increases.

Recall that according to Eeckhoudt & Schlesinger (2006), preferences are said to satisfy risk apportionment of degree \( s \) if \((-1)^{s+1} u^{(s)} \geq 0 \iff u \in U_{s-icv} \). This notion extends prudence, temperance, and edginess to any degree \( s \) and can be defined by means of comparison of specific lotteries. Here, we show that risk apportionment can be alternatively characterized by supermodularity of the expected utility viewed as a function of initial wealth \( w \) and correlation parameter \( \rho \).

We are now ready to state our main result.

**Proposition 4.1.** Assume that the decision-maker is faced with the final wealth

\[
w + (1 - I_1)X_1 + I_1 Y_1 + (1 - I_2)X_2 + I_2 Y_2
\]
where \((I_1, I_2)\) is as described in Section 2.2. The random variables \(X_1, X_2, Y_1,\) and \(Y_2\) are assumed to be mutually independent, independent from \((I_1, I_2)\), and such that \(X_1 \preceq_{s_1-icv} Y_1\) and \(X_2 \preceq_{s_2-icv} Y_2\). Then,

\[
u \in \mathcal{U}_{(s_1+s_2+1)-icv} \Rightarrow \mathcal{U}(w, \rho) \text{ is supermodular.}
\]

Conversely, if \(\mathcal{U}(w, \rho)\) is supermodular whatever \((I_1, I_2), X_1, X_2, Y_1,\) and \(Y_2\) then \(\nu \in \mathcal{U}_{(s_1+s_2+1)-icv}\).

**Proof.** The final wealth \(w + (1 - I_1)X_1 + I_1Y_1 + (1 - I_2)X_2 + I_2Y_2\) can be seen as a lottery with the following four outcomes:

\[
w + (1 - I_1)X_1 + I_1Y_1 + (1 - I_2)X_2 + I_2Y_2 = \begin{cases} 
w + X_1 + X_2 & \text{with probability } p_1p_2 + \rho, \\
w + X_1 + Y_2 & \text{with probability } p_1(1 - p_2) - \rho, \\
w + Y_1 + X_2 & \text{with probability } (1 - p_1)p_2 - \rho, \\
w + Y_1 + Y_2 & \text{with probability } (1 - p_1)(1 - p_2) + \rho. \end{cases}
\]

Let us now consider another random vector \(((1 - I'_1)X_1 + I'_1Y_1, (1 - I'_2)X_2 + I'_2Y_2)\) where \((I'_1, I'_2)\) has the same distribution as \((I_1, I_2)\), except that the correlation parameter \(\rho\) is replaced with \(\rho' > \rho\), that is,

\[
\begin{align*}
\Pr[I'_1 = 0, I'_2 = 0] &= p_1p_2 + \rho' \\
\Pr[I'_1 = 1, I'_2 = 0] &= (1 - p_1)p_2 - \rho' \\
\Pr[I'_1 = 0, I'_2 = 1] &= p_1(1 - p_2) - \rho' \\
\Pr[I'_1 = 1, I'_2 = 1] &= (1 - p_1)(1 - p_2) + \rho'.
\end{align*}
\]

Taking

\[
X = w + (1 - I'_1)X_1 + I'_1Y_1 + (1 - I'_2)X_2 + I'_2Y_2
\]

and

\[
Y = w + (1 - I_1)X_1 + I_1Y_1 + (1 - I_2)X_2 + I_2Y_2
\]

we know from Proposition 2.1 in Denuit, Eeckhoudt & Rey (2009) that \(X \preceq_{(s_1+s_2)-icv} Y\).

Invoking Proposition 2.1 ends the proof. \(\square\)

Taking \(s_1 = s_2 = 1\), and noting that \(X_i = 0 \preceq_{1-icv} a_i = Y_i\) holds for \(i = 1, 2\) (since the \(a_i\)'s are non-negative), we get the result established in Section 3.2 for prudence.
Let us discuss the meaning of Proposition 4.1. The stochastic order relation $Z_1 \preceq_{(s_1+s_2)-icv} Z_2$ expresses the preference between a pair of 4-state lotteries offering either $w + X_1 + X_2$, $w + X_1 + Y_2$, $w + Y_1 + X_2$, or $w + Y_1 + Y_2$. It means that a decision-maker with a utility function $u \in \mathcal{U}_{(s_1+s_2)-icv}$ dislikes a simultaneous increase in the probability of getting the extreme outcomes $w + X_1 + X_2$ (the worst one) and $w + Y_1 + Y_2$ (the best one) and decrease in the probability of getting the intermediate outcomes $w + X_1 + Y_2$ and $w + Y_1 + X_2$. Proposition 4.1 states that the pain caused by such a probability mass shift is decreasing in the initial wealth level $w$. Hence, the decision-maker dislikes a spread in the probabilities from the inner cases $w + X_1 + Y_2$ and $w + Y_1 + X_2$ to the outer cases $w + X_1 + X_2$ and $w + Y_1 + Y_2$.

Note that unless the correlation parameter $\rho$ controls the amount of dependence between $I_1$ and $I_2$, Proposition 4.1 does not deal with correlation aversion. Among the different terms in the final wealth, some are positively related, such as $I_1 Y_1$ and $I_2 Y_2$ or $(1 - I_1) X_1$ and $(1 - I_2) X_2$, but others are negatively related, like $(1 - I_1) X_1$ and and $I_2 Y_2$, for instance. Of course, $(1 - I_1) X_1$ and $I_1 Y_1$ are mutually exclusive (that is, only one of them can be nonzero), an extreme form of negative dependence studied in Dhaene & Denuit (1999). We will come back to this issue in the next section.

Given the importance of temperance and edginess, these concepts are discussed in details in the next sections.

### 4.2 Temperance

Temperance ($u^{(4)} \leq 0$) was first defined by Kimball (1992) in a context of risk management in the presence of background risk. A decision maker is temperant when “an unavoidable (background) risk leads him to reduce exposure to another risk even if the two risks are statistically independent”. Note that again the definition is given in the context of a specific decision problem, and not as the expression of a preference.

As it was the case for prudence, temperance also can be interpreted as an implication of the lower sensitivity to an increase in the correlation parameter $\rho$ due to an increase in initial wealth. This is a consequence of Proposition 4.1 taking $Y_i = a_i \geq 0$ for $i = 1, 2$, $X_2 = 0$ and $X_1$ independent of $(I_1, I_2)$ and such that $\mathbb{E}[X_1] \leq a_1$. The final wealth faced
by the decision-maker is $w + (1 - I_1)X_1 + I_1a_1 + I_2a_2$. Note that $(1 - I_1)X_1 + I_1a_1$ can be interpreted as a lottery giving $X_1$ with probability $p_1$ and $a_1$ with probability $1 - p_1$. Since $X_1 \preceq_{2-icv} a_1$ and $0 \preceq_{1-icv} a_2$ we are in a position to apply Proposition 4.1 with $s_1 = 2$ and $s_2 = 1$. For such $X_1$, $a_1$ and $a_2$, an increase in $\rho$ reduces welfare less for a larger value of $w$ if, and only if, the decision-maker is temperant. This means that the second mixed derivative of the expected utility

$$U(w, \rho) = \mathbb{E}[u(w + (1 - I_1)X_1 + a_1I_1 + a_2I_2)]$$

(4.1)

with respect to $w$ and $\rho$ is non-negative. Like prudence, we see that temperance is the consequence of a lower sensitivity to a change in the correlation parameter $\rho$ when wealth increases.

Another explanation for $u^{(4)} \leq 0$ comes from the sign of $\frac{\partial^2}{\partial w^2}CU_P(w, \rho)$. A DEVELOPER???

In Epstein & Tanny (1980), a correlation increasing transformation is applied to the pair $(I_1, I_2)$. It indeed increases the correlation between the variables of interest. In (4.1), increasing $\rho$ increases correlation between $a_1I_1$ and $a_2I_2$. However, increasing $\rho$ decreases the correlation between $(1 - I_1)X_1$ and $a_2I_2$ since

$$\text{Cov}[(1 - I_1)X_1, a_2I_2] = \mathbb{E}\left[\text{Cov}[(1 - I_1)X_1, a_2I_2|X_1]\right]$$

$$= -a_2\mathbb{E}[X_1]\text{Cov}[I_1, I_2] = -\rho a_2\mathbb{E}[X_1].$$

Therefore, the interpretation given here to prudence does not really refer to correlation aversion, but to a more subtle relationship between the underlying random variables as explained in Section 4.1.

4.3 Edginess

Edginess, defined by $u^{(5)} \geq 0$, was introduced by Lajeri-Chaherli (2004) in a context of multiple risks in a two-period model. Specifically, edginess captures the reactivity to multiple risks on precautionary motives. It is a necessary condition to have preferences exhibiting \textit{standard prudence or precautionary vulnerability} (we refer to Lajeri-Chaherli (2004) for
more details). Like prudence and temperance, edginess (termed as risk apportionment of order 5 in Eeckhoudt & Schlesinger, 2006) can be interpreted as the consequence of a lower sensitivity to a change in the correlation parameter ρ when wealth increases.

To illustrate this, let us consider $I_1$ and $I_2$ as defined in Section 2. Let us apply Proposition 4.1 with $Y_i = a_i \geq 0$ for $i = 1, 2$ and two independent random variables $X_1$ and $X_2$ such that $\mathbb{E}[X_i] \leq a_i$ holds for $i = 1, 2$. Then, as $X_i \preceq_{2-iwc} a_i$ is valid for $i = 1, 2$, we are in a position to apply Proposition 4.1 with $s_1 = s_2 = 2$.

For such $X_1$, $X_2$, $a_1$ and $a_2$, an increase in the correlation parameter $\rho$ reduces welfare less for a larger value of $w$, that is, the second mixed derivative of the expected utility

$$U(w, \rho) = \mathbb{E}[u(w + (1 - I_1)X_1 + a_1I_1 + (1 - I_2)X_2 + a_2I_2)] \quad (4.2)$$

with respect to $w$ and $\rho$ is non-negative. Like prudence and temperance, edginess can, thus, be defined as a lower sensitivity to a change in the correlation parameter as wealth increases.

5 Conclusion

Very often in decision problems, many results depend upon the signs of successive derivatives of the utility function. The present paper has provided new and unified interpretations of these signs. It is first shown that decision-makers whose non-decreasing utility functions has derivatives alternating in signs becomes less sensitive to detrimental changes as he gets richer. This underlies many aspects of a decision-maker’s behavior under risk, including risk aversion, prudence, temperance, and edginess. Exactly as risk aversion that has been presented from the very beginning as a form of preference independently of the context in which risk arises, the more recent notions of prudence, temperance, and edginess (and more generally the notion of risk apportionment of any degree $n$) are defined here using the idea of aversion to detrimental changes decreasing in wealth. Thus, these notions appear as natural as that of risk aversion.

This paper then considers a class of 4-state lotteries with a simple dependence structure indexed by a single correlation parameter $\rho$. Risk apportionment turns out to be equivalent to decreasing aversion to probability spreads, that is, to shifts of the probability mass from
the inner to the outer lottery outcomes. This allows us to provide a better understanding of the meaning of the sign of the successive derivatives of a utility function, complementing previous studies. Our contribution may also be adapted to experimental testing. In order to deal with general correlation increasing transformation in the sense of Epstein & Tanny (1980), we need bivariate stochastic dominance relations, as explained next.

Consider a utility functions \( u \) defined on the real plane and denote as \( u^{(i,j)} \) the \((i, j)\)th mixed partial derivative of \( u \) with respect to \( x_1 \) and \( x_2 \), that is, \( u^{(i,j)} = \frac{\partial^i+j}{\partial x_1^i \partial x_2^j} u \). Then, \((X_1, X_2)\) is said to be smaller than \((Y_1, Y_2)\) in the bivariate \((s_1, s_2)\)-increasing concave order, denoted by \((X_1, X_2) \preceq_{(s_1, s_2)-icv} (Y_1, Y_2)\), when \( E[u(X_1, X_2)] \leq E[u(Y_1, Y_2)] \) for all the utility functions \( u \) such that \((-1)^{k_1+k_2+1}u^{(k_1,k_2)} \geq 0 \) for all \( k_1 = 0, \ldots, s_1, k_2 = 0, \ldots, s_2 \), with \( k_1 + k_2 \geq 1 \). See Denuit, Eeckhoudt & Rey (2009) and the references therein for more details. For \( s_1 = s_2 = 1 \) we get the general increasing transformation of Epstein & Tanny (1980). Since the bivariante function \((x_1, x_2) \mapsto u(w + \alpha_1 x_1 + \alpha_2 x_2)\) has derivatives exhibiting the required signs whatever \( w, \alpha_1 \) and \( \alpha_2 \geq 0 \) when \( u \in U(s_1+s_2)-icv \), we have that

\[
(X_1, X_2) \preceq_{(s_1, s_2)-icv} (Y_1, Y_2) \Rightarrow w + \alpha_1 X_1 + \alpha_2 X_2 \preceq_{(s_1+s_2)-icv} w + \alpha_1 Y_1 + \alpha_2 Y_2,
\]

for all \( w, \alpha_1 \) and \( \alpha_2 \geq 0 \).

For \( s_1 = s_2 = 1 \) we get that \( w + \alpha_1 X_1 + \alpha_2 X_2 \) precedes \( w + \alpha_1 Y_1 + \alpha_2 Y_2 \) in second degree stochastic dominance.

**Acknowledgements**

Both Authors would like to express their deepest gratitude to Louis Eeckhoudt for stimulating discussions. The constructive comments and numerous suggestions from two anonymous referees and an Associate Editor have greatly helped to improve a previous version of the manuscript. In particular, they provided the correct interpretation for the shifts in the correlation parameter and suggested to move from the correlation increasing transformations to the decreasing sensitivity to detrimental changes established in Proposition 2.1.

Michel Denuit acknowledges the financial support of the Communauté française de Belgique under contract “Projet d’Actions de Recherche Concertées” ARC 04/09-320, as well
as the financial support of the Banque Nationale de Belgique under grant “Risk measures and Economic capital”.

References


