First hitting time law for some jump-diffusion processes: existence of a density

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Abstract. Let \((X_t, \ t \geq 0)\) be a diffusion process with jumps, sum of a Brownian motion with drift and a compound Poisson process. We consider \(\tau_x\) the first hitting time of a fixed level \(x > 0\) by \((X_t, \ t \geq 0)\). We prove that the law of \(\tau_x\) has a density (defective when \(E(X_1) < 0\)) with respect to the Lebesgue measure.

Keywords: Lévy process, jump-diffusion process, hitting time law.

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1 Introduction

The main purpose of this paper is to show that the first hitting time distribution associated with a jump-diffusion process (sum of a Brownian motion with drift and an independent compound Poisson process) has a density with respect to the Lebesgue measure.

Let $(X_t, t \geq 0)$ be a càdlàg process started at 0 and $\tau_x$ the first hitting time of level $x > 0$ by $X$.

Lévy, in [16], computes the law of $\tau_x$ when $X$ is a Brownian motion with drift. This result is extended by Alili, Patie and Petersen [1] or Leblanc [13] to the case where $X$ is an Ornstein-Uhlenbeck process. The case where $X$ is a Bessel process was studied by Borodin and Salminen in [4].

Some results are also available when the process $X$ has jumps. The first results are obtained by Zolotarev [23] and Borokov [5] when $X$ is a spectrally negative Lévy process. They give the law of $\tau_x$. Moreover, if $X_t$ has the probability density with respect to the Lebesgue measure $p(t, x)$, then the law of $\tau_x$ has the density with respect to the Lebesgue measure $f(t, x)$, where $xf(t, x) = tp(t, x)$. In this case where $X$ has only negative jumps, $X_{\tau_x} = x$ almost surely.

If $X$ is a spectrally positive Lévy process, Doney [7] gives an explicit formula for the joint Laplace transform of $\tau_x$ and the overshoot $X_{\tau_x} - x$. When $X$ is a stable Lévy process, Peskir [17], and Bernyk, Dalang and Peskir [2] obtain an explicit formula for the hitting time density.

The case where $X$ has signed jumps is more recently studied. In [9], the authors give the law of $\tau_x$ when $X$ is the sum of a decreasing Lévy process and an independent compound process with exponential jump sizes. This result is extended by Kou et Wang in [11] to the case of a diffusion process with jumps where the jump sizes follow a double exponential law. They compute the Laplace transform of $\tau_x$ and derive an expression for the density of $\tau_x$. For a more general jump-diffusion process, Roynette, Vallois and Volpi [20] show that the Laplace transform of $(\bar{G}_{\tau_x}, \tau_x - X_{\tau_x} - x, x - X_{\tau_x} - x, x - X_{\tau_x} - x)$ is solution of some kind of random integral.

Doney and Kyprianou [8] studied the problem for general Lévy processes. They give the quintuple law of $(\bar{G}_{\tau_x}, \tau_x - X_{\tau_x} - x, x - X_{\tau_x} - x, x - X_{\tau_x} - x, x - X_{\tau_x} - x)$ where $\bar{X}_t = \sup_{s \leq t} X_s$ and $\bar{G}_t = \sup\{s < t, \bar{X}_s = X_s\}$.

Results are also available for some Lévy processes without Gaussian component, see Lefèvre, Loisel and Picard [14, 15, 18, 19]. Blanchet [3] considers a process satisfying the following stochastic equation: $dS_t = S_t(\mu dt + \sigma \mathbf{1}_{\hat{\phi}(t)=0} dW_t + \hat{\phi}(t)=\phi d\tilde{N}_t)$, $t \leq T$ where $T$ is a finite horizon, $\mu \in \mathbb{R}$, $\sigma > 0$, $\hat{\phi}(\cdot)$ is a function taking two values 0 or $\phi$, $W$ is a Brownian motion, $N$ is a Poisson process with intensity $\frac{\sigma^2}{\phi^2} \mathbf{1}_{\hat{\phi}(t)=\phi}$ and $\tilde{N}$ is the compensated Poisson process.

The aim of our paper is to add to these studies the law of a first hitting time by a Lévy process which is the sum of a Brownian motion with drift and a compound Poisson process. We do not limit our study to a particular distribution of the jumps sizes.

This paper is organized as follows: Section 2 contains the main result (Theorem 2.1) which gives the first hitting time law by a jump Lévy process. The following two sections (Section 2.1 and Section 2.2) are dedicated to the proof of Theorem 2.1. In these sections we compute the derivative at $t = 0$ (Section 2.1) and at $t > 0$ (Section 2.2) of the hitting time distribution function. Section 3 contains the proofs of some useful results.
2 Hitting time law

Let \( m \in \mathbb{R}, (W_t, t \geq 0) \) be a standard Brownian motion, \((N_t, t \geq 0)\) a Poisson process with constant positive intensity \( a \) and \((Y_i, i \in \mathbb{N}^*)\) be a sequence of independent identically distributed random variables with distribution function \( F_{Y_i} \). We suppose that the following \( \sigma \)-fields \( \sigma(Y_i, i \in \mathbb{N}^*), \sigma(N_t, t \geq 0) \) and \( \sigma(W_t, t \geq 0) \) are independent. Let \((T_n, n \in \mathbb{N}^*)\) be the sequence of the jump times of the process \( N_t \).

Let \( X_t \) be the process defined by
\[
X_t = mt + W_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{1}
\]

Let \( \tau_x \) be the first hitting time of level \( x > 0 \) by the stochastic process \((X_t, t \geq 0)\):
\[
\tau_x = \inf \{ u > 0 : X_u \geq x \}.
\]

The main result of this paper is the following theorem. It gives the law of \( \tau_x \).

**Theorem 2.1** If there exists \( \beta > 0 \) such that \( \mathbb{E}(e^{\beta |Y_1|}) < \infty \), then the distribution function of \( \tau_x \) has a right derivative at 0 and is differentiable at every point of \( ]0, \infty[ \). The derivative, denoted \( f(., x) \), is equal to
\[
f(0, x) = a \left( 2 - F_Y(x) - F_Y(x^-) \right) + \frac{a^2}{4} \left( F_Y(x) - F_Y(x^-) \right)
\]
and for every \( t > 0 \)
\[
f(t, x) = a \mathbb{E} \left( 1_{\tau_x > t} (1 - F_Y)(x - X_t) \right) + \mathbb{E} \left( 1_{\tau_x > T_N} \tilde{f} \left( t - T_N, x - X_{T_N} \right) \right)
\]
where
\[
\tilde{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp \left[ - \frac{(z - mu)^2}{2u} \right] 1_{[0, \infty]}(u), \quad u \in \mathbb{R}, \quad z \in \mathbb{R}.
\]
Furthermore, \( \mathbb{P}(\tau_x = \infty) = 0 \) if and only if \( m + a \mathbb{E}(Y_1) \geq 0 \).

The proof of Theorem 2.1 is given in Sections 2.1 and 2.2.

Let \( \tilde{X} \) be the \( m \) drifted Brownian motion defined by
\[
\tilde{X}_t = mt + W_t, \quad \text{for all} \quad t \geq 0,
\]
and \( \tilde{\tau}_z \) be the first hitting time of \( z > 0 \) by \( \tilde{X} \). Then, according to [10], \( \tilde{f}(., z) \) is the derivative of the distribution function of \( \tilde{\tau}_z \) (see Section 3 for more details).

Let \((\mathcal{F}_t)_{t \geq 0}\) be the completed natural filtration generated by the processes \((W_t, t \geq 0)\), \((N_t, t \geq 0)\) and the random variables \((Y_i, i \in \mathbb{N}^*)\).

**Remark 2.2** This result is already known in the following cases:

- When \( X \) has no positive jumps, the law of \( \tau_x \) is given in Theorem 46.4 page 348 of [21].
- When \( X \) is a stable Lévy process, with no negative jumps, the law of \( \tau_x \) is given in [2].
- When \( X \) is a jump-diffusion where the jump sizes follow a double exponential law, the result is given in [11].
2.1 Existence of the right derivative at \( t = 0 \)

In this section we prove the first part of Theorem 2.1. Our goal is to show that the distribution function of \( \tau_x \) (i.e. \( t \mapsto \mathbb{P}(\tau_x \leq t) \)) has a right derivative at \( 0 \) and we compute this derivative.

For this purpose we split the probability \( \mathbb{P}(\tau_x \leq h) \) according to the values of \( N_h \):

\[
\mathbb{P}(\tau_x \leq h) = \mathbb{P}(\tau_x \leq h, N_h = 0) + \mathbb{P}(\tau_x \leq h, N_h = 1) + \mathbb{P}(\tau_x \leq h, N_h \geq 2).
\]

Note that \( \mathbb{P}(\tau_x \leq h, N_h \geq 2) \leq \mathbb{P}(N_h \geq 2) = 1 - e^{-ah} - ahe^{-ah} \), thus

\[
\lim_{h \to 0} \frac{\mathbb{P}(\tau_x \leq h, N_h \geq 2)}{h} = 0.
\]

The proof of the first part of Theorem 2.1 will be complete when the following lemma and proposition are shown:

Lemma 2.3 The term \( \frac{\mathbb{P}(\tau_x < h, N_h = 0)}{h} \) converges to 0 when \( h \) goes to 0.

Proposition 2.4 If there exists \( \beta > 0 \) such that \( \mathbb{E}(e^{\beta|Y_1|}) < \infty \), then for every \( x > 0 \) the term \( \mathbb{P}(\tau_x < h, N_h = 1) \) converges to \( \frac{\alpha}{2} (2 - F_Y(x)) - \frac{\alpha}{2} (F_Y(x) - F_Y(x^-)) \) when \( h \) goes to 0.

Proof of Lemma 2.3.

The essential observation is that on the set \( \{ \omega : N_h(\omega) = 0 \} \), the processes \( (X_t, 0 \leq t \leq h) \) and \( (\tilde{X}_t, 0 \leq t \leq h) \) are equal, and \( \mathbb{P} \)-almost surely \( \tau_x \wedge h = \tilde{\tau}_x \wedge h \). Since \( \tilde{\tau}_x \) is independent of \( N \), then

\[
\frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h} = \frac{e^{-ah}\mathbb{P}(\tilde{\tau}_x \leq h)}{h}.
\]

The law of \( \tilde{\tau}_x \) has a \( C^\infty \) density (possibly defective) with respect to the Lebesgue measure, null on \( (-\infty, 0) \). Thus the limit of \( \frac{\mathbb{P}(\tau_x \leq h, N_h = 0)}{h} \) exists and is equal to 0 when \( h \) goes to 0. \( \square \)

To prove Proposition 2.4, we use the same type of arguments as in [20] (for the proof of Theorem 2.4). In [20], the authors compute the joint Laplace transform of (a passage time of a Lévy process, overshoot) as solution of an integral equation.

Proof of Proposition 2.4.

We split the probability \( \mathbb{P}(\tau_x \leq h, N_h = 1) \) into three parts according to the relative position of \( \tau_x \) and \( T_1 \), the first jump time of the Poisson process \( N \):

\[
\mathbb{P}(\tau_x \leq h, N_h = 1) = \mathbb{P}(\tau_x < T_1, N_h = 1) + \mathbb{P}(\tau_x = T_1, N_h = 1) + \mathbb{P}(T_1 < \tau_x \leq h, N_h = 1).
\]

(2)

Step 1 : We prove that the contribution to the limit \( \lim_{h \to 0^+} h^{-1} \mathbb{P}(\tau_x \leq h, N_h = 1) \) of the first term on the right hand side of (2) is null.

Since the processes \( (X_t, 0 \leq t < T_1) \) and \( (\tilde{X}_t, 0 \leq t < T_1) \) are equal, then on the set \( \{ \omega, \tau_x(\omega) < T_1(\omega) \} \), the stopping time \( \tau_x \) and \( \tilde{\tau}_x \) are equal. Thus

\[
\mathbb{P}(\tau_x < T_1, N_h = 1) = \mathbb{P}(\tilde{\tau}_x < T_1 \leq h, N_h = 1) \leq \mathbb{P}(\tilde{\tau}_x \leq h).
\]
The law of $\tau_x$ has a $C^\infty$ density (possibly defective) with respect to the Lebesgue measure, null on $]-\infty, 0]$. Thus $\frac{\mathbb{P}(\tau_x < T_h, N_h = 1)}{h}$ converges to 0 when $h$ goes to 0.

**Step 2** : We prove that the contribution to the limit $\lim_{h \to 0^+} h^{-1} \mathbb{P}(\tau_x \leq h, N_h = 1)$ of the second term on the right hand side of (2) is $\frac{a}{2}(2 - F_Y(x) - F_Y(x^-))$.

Note that

$$\mathbb{P}(\tau_x = T_1, N_h = 1) = \mathbb{P}(\tau_x > T_1, X_{T_1} + Y_1 \geq x, T_1 \leq h < T_2).$$

Here, for every $n \in \mathbb{N}^*$, $T_n = S_1 + \ldots + S_n$ where $(S_i, i \geq 1)$ is a sequence of independent identically distributed random variables with exponential distribution with parameter $a$. Using the independence between $(S_i, i \geq 1)$ and $(Y_1, X, \tau_x)$ we get:

$$\mathbb{P}(\tau_x = T_1, N_h = 1) = \int_0^h a e^{-as} \int_s^\infty \mathbb{E}\left(1_{\tau_x > s}1_{Y_1 \geq x - X_{s_1}}\right) ds_2 ds_1$$

$$= a e^{-ah} \int_0^h \mathbb{E}\left(1_{\tau_x > s}1_{Y_1 \geq x - X_s}\right) ds.$$

Integrating with respect to $Y_1$, we obtain:

$$\mathbb{P}(\tau_x = T_1, N_h = 1) = a e^{-ah} \int_0^h \mathbb{E}\left((1 - F_Y)((x - \tilde{X}_s)^-)\right) ds - a e^{-ah} \int_0^h \mathbb{E}\left(1_{\tau_x \leq s}(1 - F_Y)((x - \tilde{X}_s)^-)\right) ds.$$

On the one hand, since $F_Y$ is a càdlàg bounded function and $\tilde{X}$ is a Brownian motion with drift, we get

$$\lim_{s \to 0} \mathbb{E}\left(F_Y((x - \tilde{X}_s)^-)\right) = \frac{F_Y(x) + F_Y(x^-)}{2}.$$\n
On the other hand, since the distribution function of $\tau_x$ is differentiable, then

$$\lim_{s \to 0} \mathbb{E}\left(1_{\tau_x \leq s}(1 - F_Y)((x - \tilde{X}_s)^-)\right) = 0.$$

We deduce that

$$\lim_{h \to 0} \frac{\mathbb{P}(\tau_x = T_1, N_h = 1)}{h} = \frac{a}{2}(2 - F_Y(x) - F_Y(x^-)).$$

**Step 3** : We prove that the contribution to the limit $\lim_{h \to 0^+} h^{-1} \mathbb{P}(\tau_x \leq h, N_h = 1)$ of the third term on the right hand side of (2) is $\frac{a}{4}(F_Y(x) - F_Y(x^-))$.

To this end, we state the following lemma:

**Lemma 2.5** The term $\frac{\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1)}{h}$ converges to $\frac{a}{4}(F_Y(x) - F_Y(x^-))$ when $h$ goes to 0.

**Proof** Note that

$$\mathbb{P}(T_1 < \tau_x \leq h, N_h = 1) = \mathbb{P}(T_1 < \tau_x \leq h, T_1 \leq h < T_2)$$

and $T_2 = T_1 + S_2 \circ \theta_{T_1}$ where $\theta$ is the translation operator. Here $S_2$ is a random variable with exponential distribution with parameter $a$, independent of $(T_1, (W_t, t \geq 0), (Y_i, i \in \mathbb{N}^*))$. 

5
Moreover, on \( \{ \omega : T_1(\omega) < \tau_x(\omega) \leq h < T_2(\omega) \} \), \( X_s = X_{T_1} + \tilde{X}_{s-T_1} \circ \theta_{T_1} \) when \( T_1 < s \leq h \) and \( \tau_x = T_1 + \tilde{\tau}_{x-T_1} \circ \theta_{T_1} \).

Strong Markov Property at the \( (F_t, t \geq 0) \)-stopping time \( T_1 \) gives:

\[
P(T_1 < \tau_x \leq h, \, N_h = 1) = \mathbb{E} \left( 1_{\tau_x > T_1} 1_{h > T_1} \mathbb{E}^{T_1} \left( 1_{\tilde{\tau}_{x-T_1} \leq h-T_1} 1_{h-T_1 < S_2} \right) \right)
\]

where \( \mathbb{E}^{T_1}(\cdot) = \mathbb{E}(\cdot | F_{T_1}) \).

Integrating with respect to \( S_2 \), we obtain:

\[
P(T_1 < \tau_x \leq h, \, N_h = 1) = \mathbb{E} \left( 1_{\tau_x > T_1} 1_{h > T_1} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left( 1_{\tilde{\tau}_{x-T_1} \leq h-T_1} \right) \right).
\]

Remark that \( \{ \omega : \tau_x(\omega) > T_1(\omega) \} = \{ \omega : \tilde{\tau}_x(\omega) > T_1(\omega) \} \cap \{ \omega : X_{T_1}(\omega) < x \} \). Consequently

\[
P(T_1 < \tau_x \leq h, \, N_h = 1) = -\mathbb{E} \left( 1_{\tilde{\tau}_x \leq T_1} 1_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left( 1_{\tilde{\tau}_{x-T_1} \leq h-T_1} \right) \right) + \mathbb{E} \left( 1_{h > T_1} 1_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left( 1_{\tilde{\tau}_{x-T_1} \leq h-T_1} \right) \right).
\]

Since the distribution function of \( \tilde{\tau}_x \) has a null derivative at 0, then

\[
\lim_{h \to 0} \frac{1}{h} \mathbb{E} \left( 1_{\tilde{\tau}_x \leq T_1} 1_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left( 1_{\tilde{\tau}_{x-T_1} \leq h-T_1} \right) \right) = 0.
\]

It remains to show that

\[
\lim_{h \to 0^+} \frac{G(h)}{h} = \frac{a}{4} [F(x) - F(x^-)] \quad (3)
\]

where

\[
G(h) = \mathbb{E} \left( 1_{h > T_1} 1_{X_{T_1} < x} e^{-a(h-T_1)} \mathbb{E}^{T_1} \left( 1_{\tilde{\tau}_{x-T_1} \leq h-T_1} \right) \right).
\]

Integrating with respect to \( T_1 \) and then using the fact that \( \tilde{f}(., z) \) is the derivative of the distribution function of \( \tilde{\tau}_x \), we get:

\[
G(h) = \int_0^h \int_0^{h-s} \mathbb{E} \left[ 1_{X_s + Y_1 < x} e^{-a(h-s)} \mathbb{E}^{s} \left( 1_{\tilde{\tau}_{x-Y_1} \leq h-s} \right) \right] ds
\]

\[
= ae^{-ah} \int_0^h \int_0^{h-s} \mathbb{E} \left[ 1_{\tilde{X}_s + Y_1 < x} \tilde{f}(u, x - \tilde{X}_s - Y_1) \right] duds.
\]

Since \( \tilde{X}_s = ms + W_s \), we may apply Lemma 3.1 to \( \mu = x - ms - Y_1 \) and \( \sigma = \sqrt{s} \), then

\[
G(h) = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_0^h \int_0^{h-s} \mathbb{E} \left[ e^{-\frac{(x-m(u+s)-Y_1)^2}{2(u+s)^3}} \left( \frac{x - Y_1}{(u+s)^{3/2}} + \frac{G\sqrt{s}}{\sqrt{u}(u+s)} \right) \right] duds.
\]

We make the following change of variable \( r = u + s \).

\[
G(h) = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_0^h \int_0^h \mathbb{E} \left[ e^{-\frac{(x-mr-Y_1)^2}{2r^3}} \left( \frac{x - Y_1}{r^{3/2}} + \frac{G\sqrt{s}}{r\sqrt{r-s}} \right) \right] drds.
\]
Firstly, we apply Fubini’s Theorem and secondly, we make the following change of variable $v = \frac{s}{r}$.

\[
G(h) = \frac{ae^{-ah}}{\sqrt{2\pi}} \int_{0}^{h} \int_{0}^{r} e^{-(x-Y_1)^2 / 2r^2} \left( x - Y_1 \right)^2 / r^3 + \frac{G\sqrt{v}}{r \sqrt{r-s}} + ds \, dr,
\]

where

\[
g(r) = \int_{0}^{1} e^{-(x-Y_1)^2 / 2r^2} \left( x - Y_1 \right)^2 / \sqrt{r} + \frac{G\sqrt{v}}{\sqrt{1-v}} \, dv.
\]

But,

\[
\lim_{r \to 0^+} e^{-(x-Y_1)^2 / 2r^2} \left( x - Y_1 \right)^2 / \sqrt{r} + \frac{G\sqrt{v}}{\sqrt{1-v}} = \frac{\sqrt{v}}{\sqrt{1-v}} G 1_{x=Y_1},
\]

and

\[
\sup_{0 \leq r \leq 1} e^{-(x-Y_1)^2 / 2r^2} \left( x - Y_1 \right)^2 / \sqrt{r} + \frac{G\sqrt{v}}{\sqrt{1-v}} \leq \sup_{z \geq 0} ze^{-z^2} + |m| + \frac{\sqrt{v}}{\sqrt{1-v}} |G|.
\]

Then from Lebesgue’s Dominated Convergence Theorem we obtain:

\[
\lim_{r \to 0} g(r) = \mathbb{P}(Y_1 = x) \mathbb{E}(G) \int_{0}^{1} \frac{\sqrt{v}}{\sqrt{1-v}} dv = \frac{\sqrt{2\pi}}{4} \mathbb{P}(Y_1 = x).
\]

We deduce the identity (3), i.e.

\[
\lim_{h \to 0} \frac{G(h)}{h} = \frac{a}{4} \mathbb{P}(Y_1 = x),
\]

which achieves the proof of Lemma 2.5.

Proposition 2.4 is a consequence of the Steps 1 to 3.

2.2 Existence of the derivative at $t > 0$

Our task is now to show that the distribution function of $\tau_x$ is differentiable on $\mathbb{R}_+^*$ and to compute its derivative. For this purpose we split the probability $\mathbb{P}(t < \tau_x \leq t + h)$ according to the values of $N_{t+h} - N_t$:

\[
\mathbb{P}(t < \tau_x \leq t + h) = \mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0) + \mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 1) + \mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t \geq 2).
\] (4)

The third term on the right hand side of (4) is upper bounded by

\[
\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t \geq 2) \leq \mathbb{P}(N_{t+h} - N_t \geq 2) = 1 - e^{-ah} - ahe^{-ah}.
\]
Therefore \( \lim_{h \to 0} \frac{\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0)}{h} = 0 \).

Let us study the second term on the right hand side of (4). Markov Property at \( t \) gives:
\[
\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 1) = \mathbb{E}(1_{\tau_x > t}\mathbb{P}(\tau_x - X_t \leq h, N_h = 1)),
\]
where \( \mathbb{P}(\cdot) = \mathbb{P}(\cdot|\mathcal{F}_t) \).

In virtue of Lemma 2.4, \( \mathbb{P}(\tau_x - X_t \leq h, N_h = 1) \) converges to
\[
\frac{1}{2} [2 - F_Y(x - X_t) - F_Y((x - X_t)^-)] + \frac{a}{4} [F_Y(x - X_t) - F_Y((x - X_t)^-)]
\]
and is upper bounded by \( \mathbb{P}(N_h = 1) = ae^{-ah} \leq a. \) Dominated Convergence Theorem gives:
\[
\lim_{h \to 0} \frac{\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 1)}{h} = a\mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)) + \frac{3a}{4}\mathbb{E}(1_{\tau_x > t} \Delta F_Y(x - X_t))
\]
where \( \Delta F_Y(z) = F_Y(z) - F_Y(z^-) \).

However the jumps set of \( F_Y \) (the distribution function of \( Y \)) is countable and \( X \) has a density (cf. Proposition 3.12 page 90 of [6]). Thus \( \mathbb{E}(1_{\tau_x > t} \Delta F_Y(x - X_t)) = 0 : \) indeed
\[
0 \leq \mathbb{E}(1_{\tau_x > t} \Delta F_Y(x - X_t)) \leq \mathbb{E}(1_{Y_t = -x - X_t}) = \mathbb{E}(1_{X_t = x - Y_t}) = 0.
\]

Therefore
\[
\lim_{h \to 0} \frac{\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 1)}{h} = a\mathbb{E}(1_{\tau_x > t}(1 - F_Y)(x - X_t)).
\]

The proof of the second part of Theorem 2.1 will be complete when the following proposition is shown:

**Proposition 2.6** If there exists \( \beta > 0 \) such that \( \mathbb{E}(e^{\beta|Y_t|}) < \infty \), then
\[
\lim_{h \to 0} \frac{\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0)}{h} = \mathbb{E} \left( 1_{\tau_x > T_{N_t}} \tilde{f} \left( t - T_{N_t}, x - X_{T_{N_t}} \right) \right),
\]
where \( \tilde{f} \) is the function introduced in Theorem 2.1.

**Proof** Since \( T_{N_t} \) is not a stopping time, we can not apply Strong Markov Property. We split
\[
\mathbb{P}(t < \tau \leq t + h, N_{t+h} - N_t = 0) = \sum_{k=0}^{\infty} \mathbb{P}(t < \tau \leq t + h, N_{t+h} = N_t = k)
\]
\[
= \mathbb{P}(t < \tilde{\tau}_x \leq t + h < T_1) + \sum_{k=1}^{\infty} \mathbb{P}(t < \tau \leq t + h, T_k < t < t + h < T_{k+1}).
\]

On the set \( \{(\omega, t) : T_k(\omega) < t\} \), we have \( X_t(\omega) = X_{T_k}(\omega) + X_{t-T_k} \circ \theta_{T_k}(\omega) \), hence on the set \( \{\omega, \ \tau_x(\omega) > T_k(\omega)\} \), \( \tau_x = T_k + \tau_x - X_{T_k} \circ \theta_{T_k} \). Strong Markov Property at the stopping time \( T_k \) gives
\[
\mathbb{P}(t < \tau \leq t + h, N_{t+h} - N_t = 0) = e^{-a(t+h)}\mathbb{P}(t < \tilde{\tau}_x \leq t + h)
\]
\[
+ \sum_{k=1}^{\infty} \mathbb{E}(1_{T_k < T} \mathbb{E}_{T_k} \left( 1_{t-T_k < \tilde{\tau}_x - X_{T_k} \leq t+h-T_k} 1_{t+h-T_k < S_{k+1}} \right)).
\]

On the set \( \{ (\omega, t) : \tau_z(\omega) \leq t < S_{k+1}(\omega) \} \), we have \( \tau_z = \tilde{\tau}_z \) for every \( z < 0 \). Therefore
\[
\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0) = e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_x \leq t + h)
+ \sum_{k=1}^{\infty} \mathbb{E} \left( 1_{T_k < 1} 1_{\tau_x > T_k} e^{-a(t+h-T_k)} T_k \left( 1_{t-T_k < \tilde{\tau}_x - X_{T_k} \leq t+h-T_k} \right) \right).
\]
The \( \mathcal{F}_{T_k} \)-conditional law of \( \tilde{\tau}_x - X_{T_k} \) has the density (possibly defective) \( \tilde{f}(\cdot, x - X_{T_k}) \), thus
\[
\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0) = e^{-a(t+h)} \mathbb{P}(t < \tilde{\tau}_x \leq t + h)
+ \sum_{k=1}^{\infty} \mathbb{E} \left( 1_{T_k < 1} 1_{\tau_x > T_k} e^{-a(t+h-T_k)} \int_{t-T_k}^{t+h-T_k} \tilde{f}(u, x - X_{T_k}) du \right).
\]
Let us point out that \( e^{-a(t-T_k)} = \mathbb{E}^{T_k} \left( 1_{T_{k+1} > t} \right) \) then
\[
\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0) = e^{-ah} \int_t^{t+h} \mathbb{E} \left( 1_{0 \leq t < T_k} \right) \tilde{f}(u, x) du
+ e^{-ah} \sum_{k=1}^{\infty} \int_t^{t+h} \mathbb{E} \left( 1_{T_k \leq t < k} 1_{\tau_x > T_k} \tilde{f}(u - T_k, x - X_{T_k}) \right) du,
\]
or shortly
\[
\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 0) = e^{-ah} \int_t^{t+h} \mathbb{E} \left( 1_{T_N_t < \tau_x} \tilde{f}(u - T_N_t, x - X_{T_N_t}) \right) du. \quad (5)
\]
Since \( \tilde{f} \) is continuous with respect to \( u \), then
\[
\lim_{u \to t^+} 1_{T_N_t < \tau_x} \tilde{f}(u - T_N_t, x - X_{T_N_t}) = 1_{T_N_t < \tau_x} \tilde{f}(t - T_N_t, x - X_{T_N_t}),
\]
From Lemma 3.2, Propositions 3.5 and 3.6, \( \tilde{f}(u - T_N-t, x - X_{T_N_t}) \) is dominated uniformly in \( u \) by a integrable random variable
\[
\tilde{f}(u - T_N-t, x - X_{T_N_t}) \leq c_{\varepsilon,M}(t - T_N_t)^{1-\varepsilon} \left[ \frac{1}{(x - X_{T_N_t})^{4\varepsilon}} + \exp \left( \frac{2m}{M}(x - X_{T_N_t}) \right) \right].
\]
Here \( 0 < \varepsilon < 1/4 \) and \( M > \max(1, 2|m|\beta^{-1}) \), where \( \mathbb{E}(e^{\beta|Y|}) < +\infty \) and \( c_{\varepsilon,M} \), is a constant defined in Lemma 3.2 depending only on \( \varepsilon \) and \( M \). Then, using Lebesgue’s Dominated Convergence Theorem in equation (5) we obtain
\[
\lim_{h \to 0} \frac{\mathbb{P}(t < \tau_x \leq t + h, N_{t+h} - N_t = 1)}{h} = \mathbb{E} \left( 1_{\tau_x > T_N_t} \tilde{f}(t - T_N_t, x - X_{T_N_t}) \right).
\]
\( \square \)
Using Proposition 2.6, the limit \( \lim_{h \to 0} \frac{\mathbb{P}(t < \tau_x \leq t + h)}{h} \) exists and is equal to
\[
a \mathbb{E} \left( 1_{\tau_x > t} (1 - F_Y)(x - X_t) \right) + \mathbb{E} \left( 1_{\tau_x > T_N_t} \tilde{f}(t - T_N_t, x - X_{T_N_t}) \right).
\]
The following lemma allows to conclude the proof of Theorem 2.1. The results are known, for example in [15] and [22] but for sake of completeness we give a proof.
Lemma 2.7 For all $x > 0$, the stopping time $\tau_x$ is finite almost surely if and only if $m + a\mathbb{E}(Y_1) \geq 0$.

Proof Remark that
$$\mathbb{P}(\tau_x = \infty) = \mathbb{P}(\sup_{t \geq 0} X_t < x).$$

Thanks to Theorem 7.2 page 183 of [12] which is a consequence of Strong Law of Large Numbers,
if $m + a\mathbb{E}(Y_1) > 0$, then $\lim_{t \to \infty} X_t = +\infty$ and
if $m + a\mathbb{E}(Y_1) = 0$, then $\limsup_{t \to \infty} X_t = -\liminf_{t \to \infty} X_t = \infty$.

Therefore (see Exercise 39.11 page 271 of [21]), if $m + a\mathbb{E}(Y_1) \geq 0$, then $\sup_{t \geq 0} X_t = +\infty$. This proves the first part of the lemma.

Conversely, let us suppose that $m + a\mathbb{E}(Y_1) < 0$. Then $\lim_{t \to \infty} X_t = -\infty$, and according to Theorem 48.1 page 363 of [21], $\sup_{t \geq 0} X_t < \infty$. Assume that there exists $x_0 > 0$, such that $\mathbb{P}(\tau_{x_0} < +\infty) = 1$. Then from all $x$, such that $x \leq x_0$ we have $\mathbb{P}(\tau_x < +\infty) = 1$.

Now we use recurrence reasoning. Assume that for $n \geq 1$ and for all $x \leq nx_0$, we have $\mathbb{P}(\tau_x < +\infty) = 1$. Let $x$ such that $nx_0 < x \leq (n + 1)x_0$, then using Strong Markov Property,
$$\mathbb{P}(\tau_x < +\infty) = \mathbb{E}(1_{\tau_{nx_0} < +\infty} 1_{X_{\tau_{nx_0}} \geq x}) + \mathbb{E}(1_{\tau_{nx_0} < +\infty} 1_{X_{\tau_{nx_0}} < x}\mathbb{P}^{\tau_{nx_0}}(\tau_x - X_{\tau_{nx_0}} < \infty)).$$

Using the recurrence hypothesis, since $x - X_{\tau_{nx_0}} < x_0$, almost surely $\mathbb{P}^{\tau_{nx_0}}(\tau_x - X_{\tau_{nx_0}} < \infty) = 1$ and then
$$\mathbb{P}(\tau_x < +\infty) = 1.$$

We have proved that if there exists $x_0 > 0$ such that $\mathbb{P}(\tau_{x_0} < \infty) = 1$, then for all $x > 0$, $\mathbb{P}(\tau_x < \infty) = \mathbb{P}(\sup_{t \geq 0} X_t \geq x) = 1$. This contradicts the fact that $\sup_{t \geq 0} X_t < \infty$. Then, if $m + a\mathbb{E}(Y_1) < 0$, for all $x > 0$, $\mathbb{P}(\tau_x < \infty) < 1$. \hfill $\square$

3 Appendix

A Brownian motion with drift is a process
$$\check{X}_t = mt + W_t, \quad t \geq 0$$
with $m \in \mathbb{R}$.

Let $z > 0$ and $\check{\tau}_z$ be the passage time defined by $\check{\tau}_z = \inf\{t \geq 0 : \check{X}_t \geq z\}$. By (5.12) page 197 of [10], $\check{\tau}_z$ has the following law on $\mathbb{R}_+$ :
$$\check{f}(u, z)du + \mathbb{P}(\check{\tau}_z = \infty)\delta_\infty(du)$$
where
$$\check{f}(u, z) = \frac{|z|}{\sqrt{2\pi u^3}} \exp \left[ -\frac{(z - mu)^2}{2u} \right] 1_{[0, \infty)}(u), \quad u \in \mathbb{R}, \quad \text{and} \quad \mathbb{P}(\check{\tau}_z = \infty) = 1 - e^{mz - |mz|}. \quad (6)$$

For a fixed $z$, the function $\check{f}(\cdot, z)$ and all its derivatives admit 0 as limit at 0+. The function $\check{f}$ admits an extension (denoted $\check{f}$) of class $C^\infty$ on $\mathbb{R}$, defined by $\check{f}(u, z) = 0$ for $u \leq 0$. Moreover it checks the two following lemmas :
Lemma 3.1 Let $G$ be a Gaussian random variable $\mathcal{N}(0, 1)$ and let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$. Then for every $u \in \mathbb{R}$

$$
\mathbb{E}[\tilde{f}(u, \mu + \sigma G)1_{\mu+\sigma G>0}] = \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[ e^{-\frac{(\mu - m)^2}{2(\sigma^2 + u)}} \left( \frac{\mu + \sigma^2 m}{\sqrt{u}(\sigma^2 + u)} \right) \right],
$$

where $(x)_+ = \max(x, 0)$.

**Proof** Using the probability density function of $G$ and the definition of $\tilde{f}$ (see (6)), we get:

$$
\mathbb{E}[\tilde{f}(u, \mu + \sigma G)1_{\mu+\sigma G>0}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} |\mu + \sigma g| e^{-\frac{(\mu+\sigma g-m)^2}{2u}} e^{-\frac{\sigma^2 g^2}{2}} 1_{\mu+\sigma g>0} dg.
$$

Since $|\mu + \sigma g|1_{\mu+\sigma g>0} = (\mu + \sigma g)1_{\mu+\sigma g>0} = (\mu + \sigma g)_+$, then

$$
\mathbb{E}[\tilde{f}(u, \mu + \sigma G)1_{\mu+\sigma G>0}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} (\mu + \sigma g)_+ e^{-\frac{(\mu+\sigma g-m)^2}{2u}} e^{-\frac{\sigma^2 g^2}{2}} dg.
$$

A simple computation shows that $\frac{(\mu+\sigma g-m)^2}{u} + g^2 = \frac{\sigma^2 u + \sigma^2 u}{u} + \frac{2\sigma^2 u}{\sigma^2 u}$, therefore

$$
\mathbb{E}[\tilde{f}(u, \mu + \sigma G)1_{\mu+\sigma G>0}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} (\mu + \sigma g)_+ e^{-\frac{(\mu+\sigma g-m)^2}{2u}} e^{-\frac{\sigma^2 g^2}{2}} dg.
$$

Making the following change of variable $x = \sqrt{\frac{\sigma^2 + u}{\sigma^2 + u}} \left( g + \frac{\sigma(\mu-m)}{\sigma^2 + u} \right)$, we conclude the proof

$$
\mathbb{E}[\tilde{f}(u, \mu + \sigma G)1_{\mu+\sigma G>0}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \left( \frac{\mu^2 + \sigma^2 m}{\sigma^2 + u} \right) \frac{\sigma x}{\sqrt{u}(\sigma^2 + u)} e^{-\frac{x^2}{2}} dx.
$$

\[\square\]

Lemma 3.2 Let $\tilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be the function defined by

$$
\tilde{f}(u, z) = \begin{cases} 
\frac{|z|}{\sqrt{2\pi u^3}} \exp\left[-\frac{(z-mu)^2}{2u}\right] & si \ u > 0 \\
0 & si \ u \leq 0,
\end{cases}
$$

where $m \in \mathbb{R}$. Then for every $\varepsilon > 0$ and $M \geq 1$, there exists a constant $c_{\varepsilon,M} > 0$ such that

$$
\tilde{f}(u, z) \leq c_{\varepsilon,M} u^{-1+\varepsilon} \left( \frac{1}{|z|^{1+\varepsilon}} + \exp\left(\frac{2mz}{M}\right) \right).
$$

**Proof** Let $\varepsilon > 0$ and $M \geq 1$ be fixed. Remark that it is enough to prove that there exists a constant $\tilde{c}_{\varepsilon,M} > 0$ such that

$$
\tilde{f}(u, z) \leq \tilde{c}_{\varepsilon,M} u^{-1+\varepsilon} \frac{1}{|z|^{2\varepsilon}} \exp\left(\frac{mz}{M}\right).
$$
Then we conclude the proof using the inequality $x_1x_2 \leq \frac{x_1^2 + x_2^2}{2}$ for $x_1 = \frac{1}{|\varepsilon|^2}$ and $x_2 = \exp \left( \frac{mz}{M} \right)$.

We now seek to upper bound the quotient $\frac{\int f(u, z)}{u^{-1+\varepsilon} \frac{1}{|\varepsilon|^2 \sqrt{2\pi} \exp \left( \frac{mz}{M} \right)}} \leq 0$, $u \in \mathbb{R}_+$.

Since $\exp \left( -\frac{(z-mu)^2}{2uM} \right) \leq \exp \left( -\frac{(z-mu)^2}{2uM} \right)$, then

$$\frac{\int (u, z)}{u^{-1+\varepsilon} \frac{1}{|\varepsilon|^2 \sqrt{2\pi} \exp \left( \frac{mz}{M} \right)}} \leq \frac{\left[ \frac{|z|^2}{2uM} \exp \left( -\frac{(z-mu)^2}{2uM} \right) \right]}{u^{-1+\varepsilon} \frac{1}{|\varepsilon|^2 \sqrt{2\pi} \exp \left( \frac{mz}{M} \right)}}$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{z^2}{u} \right)^{\frac{1}{2}+\varepsilon} \exp \left[ -\frac{z^2}{2uM} \right] \exp \left[ -\frac{m^2u}{2M} \right]$$

$$\leq M^2 + 2\varepsilon \frac{1}{\sqrt{2\pi}} \left( \frac{z^2}{2uM} \right)^{\frac{1}{2}+\varepsilon} \exp \left[ -\frac{z^2}{2uM} \right].$$

Since the function $x \mapsto x^{\frac{1}{2}+\varepsilon}e^{-x}$ is continuous, null at 0 and at $+\infty$, then it is bounded on $\mathbb{R}_+$. Hence there exists $c_2 > 0$ such that $x^{\frac{1}{2}+\varepsilon}e^{-x} \leq c_2$ for any $x \in \mathbb{R}_+$. We apply this inequality to $x = \frac{z^2}{2uM}$, so $\left( \frac{z^2}{2uM} \right)^{\frac{1}{2}+\varepsilon} \exp \left[ -\frac{z^2}{2uM} \right] \leq c_2$ and the proof is complete. \hfill \Box

Lemma 3.3 Let $G$ be a Gaussian random variable $\mathcal{N}(0, 1)$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $0 < \alpha < 1$. Then there exists two constants $k_{1,\alpha} > 0, k_{2,\alpha} > 0$, depending only on $\alpha$, such that

$$\mathbb{E}(\|\mu + \sigma G\|^{-\alpha}) \leq \frac{k_{1,\alpha}}{\sigma} \|\mu\|^{-\alpha} + k_{2,\alpha} \sigma^{-\alpha}.$$

**Proof** Using the probability density function of $G$, we get :

$$\mathbb{E}(\|\mu + \sigma G\|^{-\alpha}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (\mu + \sigma x)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \mu x + \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} (-\mu - \sigma x)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx.$$

Integration by parts gives :

$$\mathbb{E}(\|\mu + \sigma G\|^{-\alpha}) = \left[ \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} \frac{(\mu + \sigma x)^{1-\alpha}}{\sigma(1-\alpha)} \right]_{-\frac{\mu}{\sigma}}^{\infty} + \frac{1}{\sigma(1-\alpha)} \int_{-\frac{\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} (\mu + \sigma x)^{1-\alpha} xe^{-\frac{x^2}{2}} dx$$

$$+ \left[ \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}} \frac{(-\mu - \sigma x)^{1-\alpha}}{-\sigma(1-\alpha)} \right]_{-\infty}^{-\frac{\mu}{\sigma}} - \frac{1}{\sigma(1-\alpha)} \int_{-\infty}^{-\frac{\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} (-\mu - \sigma x)^{1-\alpha} xe^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sigma(1-\alpha)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} |\mu + \sigma x|^{1-\alpha} e^{-\frac{x^2}{2}} (x1_{x > -\frac{\mu}{\sigma}} - x1_{x \leq -\frac{\mu}{\sigma}}) dx$$

$$= \frac{1}{\sigma(1-\alpha)} \mathbb{E} \left[ |\mu + \sigma G|^{1-\alpha} G(1_{G > -\frac{\mu}{\sigma}} - 1_{G \leq -\frac{\mu}{\sigma}}) \right].$$

However for every $\mu \in \mathbb{R}$ and $\sigma > 0$, we have

$$|G(1_{G > -\frac{\mu}{\sigma}} - 1_{G \leq -\frac{\mu}{\sigma}})| \leq |G| - 2G1_{G \leq 0} = |G|.$$

Consequently $\mathbb{E}(\|\mu + \sigma G\|^{-\alpha}) \leq \frac{1}{\sigma(1-\alpha)} \mathbb{E}(\|G\| |\mu + \sigma G|^{1-\alpha}).$

The inequality $|\mu + \sigma G|^{1-\alpha} \leq (|\mu| + |\sigma|G)^{1-\alpha} \leq |\mu|^{1-\alpha} + \sigma^{1-\alpha}|G|^{1-\alpha}$ concludes the proof. \hfill \Box
Lemma 3.4 For every \( t > 0 \) and \((\alpha, \gamma) \in ]1, \infty[^2\), the following two series
\[
\sum_{i \geq 1} \mathbb{E} (1_{t>T_i} (t - T_i)^\alpha T_i^\gamma) z^i \\
\sum_{i \geq 1} \mathbb{E} (1_{t>T_i} (t - T_i)^\alpha T_i^\gamma) iz^i,
\]
have an infinite radius of convergence.

Proof It is enough to prove that the first series has an infinite radius of convergence.

Note that for \( i \geq 1 \), \( T_i \) admits as density the function \( u \mapsto u^{i-1} e^{-au} \), thus
\[
\mathbb{E} (1_{t>T_i} (t - T_i)^\alpha T_i^\gamma) = \frac{a^i}{(i-1)!} \int_0^t e^{-au} (t-u)^{\gamma+i-1} du
\leq \frac{a^i}{(i-1)!} \int_0^t (t-u)^{\gamma+i-1} du
= \frac{a^i}{(i-1)!} t^{\gamma+i} \frac{\Gamma(\gamma+i)\Gamma(\alpha+1)}{\Gamma(\gamma+i+\alpha+1)}
\]
and the conclusion holds.

As a consequence of Lemma 3.4, we have the following two propositions which are used to prove Theorem 2.1.

Proposition 3.5 Assume that there exists \( \beta > 0 \) such that \( \mathbb{E} (e^{\beta|Y_1|}) < \infty \). For every \( t > 0 \), \( \varepsilon > 0 \) and \( M \geq 1 \) such that \( \frac{2\varepsilon M}{M} \leq \beta \), the random variable \((t - T_{N_i})^{-1+\varepsilon} \exp \frac{2m}{M}(x - X_{T_{N_i}})\) \( \mathbb{P} \)-integrable.

Proof Note that
\[
(t - T_{N_i})^{-1+\varepsilon} \exp \frac{2m}{M}(x - X_{T_{N_i}}) = t^{-1+\varepsilon} \exp \frac{2mx}{M} \mathbf{1}_{t<T_i} + \sum_{i=1}^\infty \mathbf{1}_{T_i \leq t < T_{i+1}} (t - T_i)^{-1+\varepsilon} \exp \frac{2m}{M} (x - X_{T_i}),
\]
\[
\leq t^{-1+\varepsilon} \exp \frac{2mx}{M} + \sum_{i=1}^\infty \mathbf{1}_{t>T_i} (t - T_i)^{-1+\varepsilon} \exp \left[ \frac{2m}{M} (x - X_{T_i}) \right].
\]

Let \( \sigma(T_i) \) be the \( \sigma \)-field generated by \( T_i, i > 0 \).
Conditioning by \( T_i \), we obtain that
\[
\mathbb{E} \left[ \exp \left[ \frac{2m}{M} (x - X_{T_i}) \right] \big| \sigma(T_i) \right] = e^{\frac{2mx}{M} - \frac{2m^2}{M^2} T_i + \frac{2m^2}{M^2} T_i \exp \left( -\frac{2m}{M} Y_1 \right)^i}.
\]
Since \( M \geq 1 \), then \( e^{-\frac{2m^2}{M^2} T_i} \leq 1 \), therefore
\[
\mathbb{E} \left[ \exp \left[ \frac{2m}{M} (x - X_{T_i}) \right] \big| \sigma(T_i) \right] \leq e^{\frac{2m}{M} x} \mathbb{E} \left( e^{-\frac{2m}{M} Y_1} \right)^i.
\]
Consequently,
\[ \mathbb{E} \left[ (t - T_{N_i})^{-1+\varepsilon} \exp \left[ \frac{2m}{M} (x - X_{T_{N_i}}) \right] \right] \leq e^{\frac{2m}{M}x} \sum_{i=1}^{\infty} \mathbb{E} [1_{t>T_i} (t - T_i)^{-1+\varepsilon}] \mathbb{E} \left( e^{-\frac{2m}{M} Y_1} \right)^i. \]

By the choice of $M$, $\mathbb{E}(e^{-\frac{2m}{M} Y_1}) < \mathbb{E}(e^{\beta |Y_1|}) < \infty$. We use Lemma 3.4, and the conclusion holds. \hfill \square

Another useful result for the proof of Theorem 2.1 is the following:

**Proposition 3.6** Assume that there exists $\beta > 0$ such that $\mathbb{E}(e^{\beta |Y_1|}) < \infty$. For every $t > 0$ and $0 < \varepsilon < \frac{1}{4}$ the random variable $(t - T_{N_i})^{-1+\varepsilon}|X_{T_{N_i}} - x|^{-4\varepsilon}$ is $\mathbb{P}$-integrable.

**Proof** Note that
\[
(t - T_{N_i})^{-1+\varepsilon}|X_{T_{N_i}} - x|^{-4\varepsilon} = t^{-1+\varepsilon}|x|^{-4\varepsilon} 1_{t<T_1} + \sum_{i=1}^{\infty} 1_{T_i \leq t < T_{i+1}} (t - T_i)^{-1+\varepsilon}|X_{T_i} - x|^{-4\varepsilon},
\]
\[
\leq |x|^{-4\varepsilon} t^{-1+\varepsilon} + \sum_{i=1}^{\infty} 1_{T_i \leq t} (t - T_i)^{-1+\varepsilon}|X_{T_i} - x|^{-4\varepsilon}.
\]

We apply Lemma 3.3 to $\alpha = 4\varepsilon$, $G = \frac{W_{T_i}}{\sqrt{T_i}}$, $\mu = mT_i + \sum_{j=1}^{i} Y_j - x$ and $\sigma = \sqrt{T_i}$. There exists $k_{1,\varepsilon} > 0$, $k_{2,\varepsilon} > 0$ such that
\[
\mathbb{E} \left[ |X_{T_i} - x|^{-4\varepsilon} \right] \leq \frac{k_{1,\varepsilon}}{\sqrt{T_i}} |mT_i + \sum_{j=1}^{i} Y_j - x|^{-4\varepsilon} + k_{2,\varepsilon} T_i^{-4\varepsilon}.
\]

Let us use the inequality $x_1^{-\alpha} \leq 1 + x_1$, $0 < \alpha < 1$, $x_1 \geq 0$ for $x_1 = |mT_i + \sum_{j=1}^{i} Y_j - x|$ and $\alpha = 4\varepsilon$ :
\[
\mathbb{E} \left[ |X_{T_i} - x|^{-4\varepsilon} \sigma(T_i, Y_j, j \leq i) \right] \leq k_{1,\varepsilon} T_i^{-\frac{1}{2}} + k_{1,\varepsilon} T_i^{-\frac{1}{2}} |mT_i + \sum_{j=1}^{i} Y_j - x| + k_{2,\varepsilon} T_i^{-4\varepsilon}.
\]

Since $|mT_i + \sum_{j=1}^{i} Y_j - x| \leq |mT_i + \sum_{j=1}^{i} Y_j| + |x|$, then
\[
\mathbb{E} \left[ 1_{T_i < t} \frac{(t - T_i)^{-1+\varepsilon}}{|X_{T_i} - x|^{-4\varepsilon}} \right] \leq (k_{1,\varepsilon} + |x|) \mathbb{E} \left[ 1_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-\frac{3}{2}} \right] + k_{1,\varepsilon} |m| \mathbb{E} \left[ 1_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-\frac{3}{2}} \right] + k_{1,\varepsilon} \mathbb{E} \left[ |Y_i| \right] \mathbb{E} \left[ 1_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-\frac{3}{2}} \right] + k_{2,\varepsilon} \mathbb{E} \left[ 1_{T_i < t} (t - T_i)^{-1+\varepsilon} T_i^{-4\varepsilon} \right].
\]

We conclude the proof using Lemma 3.4. \hfill \square
References


