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Abstract

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1 Introduction

It is now well recognized that financial distributions returns are generally not Gaussian, present asymmetries and have fat tails. For a survey of the literature see Cont [2001] and for a review on stock return volatility see (Campbell, Lo, and MacKinlay 1997). To take into account these observed facts, the answer is to develop models which incorporate stochastic volatility or jumps, eventually both. This problem has been very soon identified in market options trading and the volatility smile appears to the traders when they began to use the Black and Scholes model. Many researchers suggested appropriate models, for example Hull and White (1987), Wiggins (1987), Stein and Stein (1991), further Bakshi, Cao and Chen (1997) and many others. The most popular stochastic volatility model is the Heston (1993) model where the volatility is modeled by a square root process introduced in finance by Cox Ingersoll and Ross (1985) in a very general context. The literature on continuous time portfolio selection is huge. The more general problem of finding the optimal couple consumption investment has deeply been investigated. The Merton (1971) paper is the cornerstone in the field. In this seminal work Merton considers a general diffusion process for the financial asset prices where instantaneous mean returns and volatilities depend on the financial prices of all the assets in the market, thus considering stochastic volatilities de facto. In a general paper Richard (1979) develops a generalized asset pricing model. In this article he considers the mean instantaneous returns and the volatilities depend on a set of state variables which follow a multidimensional diffusion process. In these general models the investment consumption problem is
partially solved. Using a dynamic programming approach the optimal portfolio weights are given up to the solution of the so called Hamilton Jacobi Bellman equation. In general this non linear partial differential equation can not be solved in closed form, and it is only in very particular cases that explicit solutions are available. It is of course possible when the assets prices follow geometric Brownian motions, Merton analyses in depth this case and exhibits many interesting properties. Texts on portfolio selection focussing on stochastic volatility are rare. Nevertheless we can quote three important contributions. In a chapter of their book, ‘Derivatives in financial markets with stochastic volatility’, Fouque, Papanicolaou, and Sircar (2000) consider the volatility is a function of a state variable which follows an Ornstein Uhlenbeck process. Chacko and Viceira model the price process such that the proportional changes in volatility follow a mean-reverting, square-root process. They obtain an exact solution in the case of unit elasticity of intertemporal substitution of consumption and approximate solution for their general case. Liu, as Richard, models prices such that the instantaneous mean returns and volatilities depend on state variables, when restricting theses characteristics to quadratic processes and using constant relative risk aversion utilities he obtains quasi explicit solutions. As a particular case, he solves in closed form solution the portfolio selection problem with the Heston’s volatility choice. In our paper we keep the idea of a model with a state variable: the volatility, so we are in line with Richard and Liu, however the model we choose for the volatility is not a quadratic function of itself and can not be embedded in Liu’s model. We choose to model the volatility such that its logarithm follows an Ornstein Uhlenbeck process. The mean reverting property is well documented for volatility (Cf. Campbell, Lo, and MacKinlay (1997), Fouque, Papanicolaou, and Sircar (2000)) and this model fits well to actual market data. We develop and ad hoc methodology. We show that the Hamilton Jacobi Bellman can be solved numerically. So we can give an in depth analysis of the impact of volatility in asset allocation and optimal consumption problems. The paper is organized as follows. In section 1, we state the problem and give the general framework of our analysis, defining the dynamics of the financial assets, discuss the choice of the stochastic volatility modelling, and give the optimal policies. In section 2 we used a CRRA utility function, analyze the optimal growth of wealth, suggest a solution for the HJB equation and illustrate the results in a numerical simulation. A conclusion ends the article. Proofs of main results are given in appendix.

2 The Investment and Consumption Intertemporal Problem: Generalities

We consider an economic agent who plans to consume and invest in a financial market during the fixed time period \([0, T]\). The financial market is perfect and contains two assets: a risk free asset with a constant return \(r\) which represents the interest rate in the economy, and a risky asset whose dynamics will be defined later. The investor consumes at a rate \(c\) during the period \([0, T]\) and optimally manages the couple consumption-investment. We consider three cases,

(i) the economic agent maximizes the aggregate utility of consumption over time and furthermore takes into account the utility of his terminal wealth;

(ii) he maximizes the aggregate utility of his consumption over time without taking any care of his terminal wealth level;

(iii) he only considers the utility of his terminal wealth, this last case is a pure portfolio selection problem.

In this section we formally define the problem, then we give the general form of the solution and suggest an algorithm to solve the Hamilton Jacobi Bellman equation associated with the problem. We finally
examine the stochastic volatility impact on consumption and investment.

2.1 The Setup

The financial asset prices obey the following equations

\[
\begin{align*}
    dS_0(t) &= rS_0(t)dt, \\
    dS_1(t) &= \mu S_1(t)dt + V(t)S_1(t)dZ_1(t), \\
    dV(t) &= \left(\kappa(\alpha - \ln V(t)) + \frac{\sigma^2}{2}\right) V(t)dt + \sigma V(t)dZ_2(t),
\end{align*}
\]

where \( S_0 \) is the riskless asset with constant return \( r \) and \( S_1 \) is the risky asset (stocks) with instantaneous return \( \mu \) and stochastic volatility \( V(t) \). The processes \( Z_1 \) and \( Z_2 \) are two correlated standard Brownian motions: \( \text{E}[dZ_1(t) \cdot dZ_2(t)] = \rho dt \). Notice that we can recover the classical constant volatility model by setting \( \sigma = 0 \) and \( \kappa = 0 \) and remark that

\[
\text{corr}\left(\frac{dS_1}{S_1}, dV\right) = \rho dt.
\]

Empirical studies show that financial asset prices are decreasing function of their volatility. They tend to go down when volatility goes up. This phenomenon is known as the leverage effect (Black 1976, Christie 1982). It is the reason why the coefficient \( \rho \) is generally found to be negative.

Applying Ito’s lemma to equation (3), the dynamics of the log-volatility \( \ln V \) reads

\[
d(\ln V(t)) = \kappa(\alpha - \ln V(t))dt + \sigma dZ_2(t).
\]

It shows that, as in Scott (1987), the log-volatility follows an Ornstein-Uhlenbeck process, where \( \kappa \) is the rate of mean reversion, \( \alpha \) the long-run mean level of \( \ln V \) and \( \sigma \) the diffusion coefficient. This model fits well market data (Masoliver and Perello 2006) and avoids the major problem encountered with Stein and Stein (1991) model which allows for negative values of the volatility. Alternative choices that ensure the positivity of the volatility are provided by Hull and White (1987) who consider that the squared volatility follows a Geometric Brownian motion or by Heston (1993) who focuses on a mean reverting square root process. In the context of optimal asset allocation with stochastic volatility, this latter case has already been investigated by Chacko and Viceira (2005) and Liu (2007).

By integration of equation (5), we obtain

\[
\ln V(t) = \alpha + e^{-\kappa t}(\ln V(0) - \alpha) + \sigma \int_0^t e^{-\kappa(t-s)} dZ_2(s),
\]

from which we get

\[
\begin{align*}
    \text{E}[V(t)] &= e^{\alpha} + e^{-\kappa t}(\ln V(0) - \alpha) + \frac{\sigma^2(1-e^{-2\kappa t})}{2\kappa}, \\
    \text{Var}[V(t)] &= \left(e^{\frac{\sigma^2(1-e^{-2\kappa t})}{2\kappa}} - 1\right) e^{2(\alpha + e^{-\kappa t}(\ln V(0) - \alpha)) + \frac{\sigma^2(1-e^{-2\kappa t})}{2\kappa}}.
\end{align*}
\]

We note that \( \ln V(t) \) has a long run Gaussian law \( \mathcal{N}\left(\alpha, \frac{\sigma^2}{2\kappa}\right) \). We also remark the long term mean value of volatility \( (t \to \infty) \) reads \( e^{\alpha + \frac{\sigma^2}{4\kappa}} \). If we want it to be equal to a constant volatility \( \sigma_s \) that we consider to be representative of the long term volatility, we get the relation:

\[
\alpha = \ln \sigma_s - \frac{\sigma^2}{4\kappa}.
\]
The mean value (7) depends on time \( t \) and on parameters \( \sigma \) and \( \kappa \). The variance of the volatility (8) also depends on these parameters. It is an increasing function of \( \sigma \), and a decreasing function of \( \kappa \). The larger \( \sigma \) the larger the fluctuations of the volatility around its long term means. Hence, \( \sigma \) captures a potential risk and more precisely the uncertainty about the future level of the volatility, i.e. the future level of the realized risk. Besides, according to (7), the smaller \( \kappa \), the more the volatility is expected to remain close to its current level. Therefore, when \( \kappa \) is small enough, i.e. when the mean reversion is slow, the volatility is expected to remain high when it is presently high and to remain low when it is currently low. On the contrary, when \( \kappa \) is large, i.e. when the mean reversion is fast, the volatility is expected to decline when it is presently high (above its long term mean) and to increase when it is currently low (below its long term mean).

We denote by \( c(t) \) the investor’s consumption rate, and by \( w(t) \) the fraction of wealth invested in the risky asset. As usual, we assume that (i) there are no transaction costs, taxes, or asset indivisibility; (ii) the agent is a price taker; (iii) short sales of all assets, with full use of proceeds, are allowed; and (iv) the agent’s portfolio finances his (her) consumption. The wealth dynamics is then given by

\[
dW(t) = \left[ (\mu - r)w(t)W(t) + rW(t) - c(t) \right] dt + w(t)W(t)V(t)dZ_1(t). \tag{10}
\]

The investor-consumer maximizes the expected utility

\[
\max_{c(t), w(t)} \mathbb{E} \left[ \int_0^T \theta e^{-\lambda s}u(c(s))ds + (1 - \theta)e^{-\lambda T}B(W(T)) \right], \tag{11}
\]

where \( \lambda \geq 0 \) is the subjective discount rate and \( \theta \in [0, 1] \) determines the relative importance of the intermediate consumption and the bequest, as in Liu (2007). When \( \theta = 1 \), the expected utility only depends on the terminal wealth and we get a pure portfolio selection problem.

We choose a HARA utility function for the consumption

\[
u(c) = \begin{cases} 
\frac{1 - \gamma}{\gamma} \left( \frac{\beta c}{1 - \gamma} + \eta \right) \gamma, & \gamma < 1, \gamma \neq 0 \\
\ln(\beta c + \eta), & \gamma = 0
\end{cases} \tag{12}
\]

with \( \beta > 0, \eta \in \mathbb{R} \), and

\[
c > K := \frac{(\gamma - 1)\eta}{\beta}, \tag{13}
\]

where \( K \) denotes the minimum subsistence level. A CARA utility function is used for the bequest

\[
B(W) = \begin{cases} 
\frac{W\gamma}{\gamma}, & \gamma < 1, \gamma \neq 0 \\
\ln W, & \gamma = 0
\end{cases} \tag{14}
\]

This latter choice is mainly motivated by the tractability it offers to the calculations.

### 2.2 Optimal policies

Now that we have defined the general framework of our model, we use the classical dynamic programming approach to solve it and introduce the indirect utility function \( J(W, V, t) \)

\[
J(W, V, t) = \max_{c(t), w(t)} \mathbb{E}_t \left[ \int_t^T \theta e^{-\lambda s}u(c(s))ds + (1 - \theta)e^{-\lambda T}B(W(T)) \right], \tag{15}
\]
where $E_t[\cdot]$ is the conditional expectation operator given information up to time $t$, i.e. the filtration generated by the Brownian motions $Z_1$ and $Z_2$. We note $V = V(t)$ and $W = W(t)$. The principle of optimality leads to the Hamilton-Jacobi-Bellman equation

$$
0 = \max_{c, w} \left\{ \theta e^{-\lambda t} u(c) + J_t + [(\mu - r)wW + rW - c] \\
+ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] J_V + \frac{\sigma^2}{2} V^2 J_{VV} \\
+ \frac{1}{2} w^2 W^2 V^2 J_{WW} + \rho \sigma w W^2 V J_{VW} \right\},
$$

with the boundary condition

$$
J(W, V, T) = (1 - \theta)e^{-\lambda T} B(W).
$$

The subscripts $W$, $V$ and $t$ are used to denote the partial derivative with respect to the corresponding variable. In the sequel, we use the superscript $*$ to denote an optimal solution: portfolio weight in the risky asset $w^*$ and consumption $c^*$.

### 2.2.1 Optimum consumption and asset allocation

We start with the case with intermediate consumption $\theta \neq 0$. The first order conditions for equation (16) are

$$
c^* = u^{-1} \left( \frac{e^{\lambda t} J_W}{J_V} \right),
$$

$$
w^* = - \frac{J_W}{W J_{WW} V(t)^2} (\mu - r) - \frac{J_{W V}}{W J_{WW}^2 \rho \sigma}.
$$

These are the usual generic formulae. Introducing $K'(V) := \frac{\lambda - \gamma r}{1 - \gamma} - \frac{\gamma(\mu - r)^2}{2 (1 - \gamma)^2 V^2}$, the calculations can go further. We prove in the Appendix that

**Proposition 1.** In the stochastic volatility model defined by (1-3), with the criterion (11) and $\theta \in (0, 1]$ the optimal consumption and portfolio rules are given by

- Case $\gamma \neq 0$:
  $$
c^*(W, V, t) = K + (1 - \gamma) \left( \frac{\theta \beta \gamma}{|\gamma|} \right)^{\frac{1}{1 - \gamma}} \frac{W}{F(V, t)} \left( 1 - \frac{K}{W} \cdot \frac{1 - e^{-r(T-t)}}{r} \right),
  $$
  $$
w^*(W, V, t) = \left( \frac{\mu - r}{(1 - \gamma) V^2} + \rho \sigma \frac{F_V(V, t)}{F(V, t)} \right) \left( 1 - \frac{K}{W} \cdot \frac{1 - e^{-r(T-t)}}{r} \right),
  $$

where $F(V, t)$ is the solution to the PDE

$$
(1 - \gamma) \left[ \frac{\theta \beta \gamma}{|\gamma|} \right]^{\frac{1}{1 - \gamma}} = K'(V) F - F_t - \left\{ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] + \frac{\gamma}{1 - \gamma} (\mu - r) \rho \sigma \right\} F_V \\
- \frac{\sigma^2}{2} V^2 F_{VV} + \gamma \frac{\sigma^2}{2} (1 - \rho^2) V^2 \frac{F^2}{F},
$$

with the boundary condition $F(V, T) = \left( \frac{1 - \theta}{|\gamma|} \right)^{\frac{1}{1 - \gamma}}$.
• Case $\gamma = 0$:

$$
\begin{aligned}
\begin{cases}
e^* &= K + \left[ \frac{1}{\lambda} + \left( \frac{1-\theta}{\theta} - \frac{1}{\lambda} \right) e^{-\lambda(T-t)} \right]^{-1} \cdot \left( W - K \cdot \frac{1-e^{-r(T-t)}}{r} \right), \\
w^* &= \frac{\mu - r}{\sqrt{V}} \left( 1 - \frac{K}{W} \cdot \frac{1-e^{-r(T-t)}}{r} \right).
\end{cases}
\end{aligned}
$$

(24)

(25)

The result stated in proposition 1 holds for all $\gamma < 1$ including $\gamma = 0$. In this latter case, HJB equation can be solve analytically. In all other cases, there is no hope to solve this non-linear PDE in closed form. A numerical solution based on a recursive scheme will be given latter on.

It results from the proof of proposition 1 that $F(V,t)$ is a positive function, and since the rightmost term in (21) remains positive along any wealth trajectory\(^1\), it ensures that the consumption is always larger than the minimum subsistence level $K$. When $\gamma \neq 0$, the optimal weight $w^*$ exhibits the additional term $\rho \sigma \frac{F_V(V,t)}{F(V,t)}$ which can be interpreted as an intertemporal hedging correction. Equations (22) and (25) show that this additional term is zero when the volatility is deterministic ($\sigma = 0$), when the correlation $\rho$ between the risky asset and the volatility is null so that no hedging of the volatility risk is possible, or when investors are myopic ($\gamma = 0$). Notice that while the demand for the risky asset is always positive when the investor is myopic provided that the risk premium is positive ($\mu > r$), the demand for the risky asset may be negative in the presence of stochastic volatility due to the presence of the term of hedging. Besides, proposition 1 shows that the consumption decreases when the function $F$ increases, namely $\partial_V c^*$ and $F_V$ have opposite signs. As a consequence, considering the realistic case $\rho < 0$, we can conclude that if the consumption decreases when the volatility increases – i.e. $\partial_V c^* < 0$ – the demand for risky asset is always lesser for an investor characterized by $\gamma \neq 0$ than for a myopic investor since the term of hedging $\rho \sigma \frac{F_V(V,t)}{F(V,t)}$ is negative.

As a corollary, and for comparison purpose, setting $\kappa = \sigma = 0$ in equation (3), we recover the case of a constant volatility, i.e. the risky asset follows a Geometric Brownian motion with volatility $\sigma_s$, say. It extends on Merton (1971) example that only considers the case $\theta = 1$, i.e. the absence of bequest.

Corollary 1. In the constant volatility model defined by (1-3) with $\kappa = \sigma = 0$, and the criterion (11) with $\theta \in (0,1)$, the optimal consumption and portfolio rules are given by

$$
\begin{aligned}
\begin{cases}
c^* &= K + \frac{W}{F(t)} \cdot \left( 1 - \frac{K}{W} \cdot \frac{1-e^{-r(T-t)}}{r} \right), \\
w^* &= \frac{\mu - r}{(1-\gamma)\sigma_s^2} \left( 1 - \frac{K}{W} \cdot \frac{1-e^{-r(T-t)}}{r} \right),
\end{cases}
\end{aligned}
$$

(26)

(27)

where

$$
F(t) = \frac{1}{K'(\sigma_s)} + \left[ \frac{\beta^{-\gamma}}{\gamma - 1} \left( \frac{1-\theta}{\theta} \right)^{-\gamma} - \frac{1}{K'(\sigma_s)} \right] e^{-K'(\sigma_s)(T-t)}.
$$

(28)

Proposition 1 and corollary 1 show that the optimal consumption has the same functional form with or without stochastic volatility. Again, the demand for the risky asset is always positive when the volatility is constant provided that the risk premium is positive ($\mu > r$).

\(^1\)By substitution of (21) and (22) into (10), we can easily conclude that the stochastic process $Y_t = W_t - K \cdot \frac{1-e^{-r(T-t)}}{r}$ remains positive provided that $Y_0 > 0$, i.e. $W_0 > K \cdot \frac{1-e^{-r(T)}}{r}$. 

6
2.2.2 Pure portfolio case

With \( \theta = 0 \), we recover the portfolio selection problem, whose solution is obtained through the maximization of the expected utility of the terminal wealth

\[
\max_{w(t)} E \left[ e^{-\lambda T} B(W(T)) \right].
\]  

Using the previous general result, we prove in appendix that

**Proposition 2.** In the pure portfolio case, with the stochastic volatility model defined by (1-3) and the criterion (11) with \( \theta = 0 \), the optimal fraction of wealth invested in the risky asset is

\[
w^*(V, t) = \frac{\mu - r}{(1 - \gamma)V^2} + \rho \sigma \frac{F_V(V, t)}{F(V, t)},
\]

where \( F(V, t) \) solves the PDE:

\[
0 = K'(V)F - F_t - \left\{ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] + \frac{\gamma}{1 - \gamma}(\mu - r)\rho \sigma \right\} F_V
\]
\[
- \frac{\sigma^2}{2}V^2F_{VV} + \gamma \frac{\sigma^2}{2}V^2(1 - \rho^2)\frac{F^2}{F},
\]

with the boundary condition \( F(V, T) = |\gamma|^{-\frac{1}{1-\gamma}}, \forall V \).

**Remark 1.** We can notice that equation (31) is the homogeneous PDE associated with (23), i.e. the PDE for \( F \) in the case with intermediate consumption. Setting \( F(V, t) := H(V, T)^{\frac{1-\gamma(1-\rho^2)}{1-\gamma}}, \) equation (31) leads to the linear second order PDE

\[
H_t + \left\{ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] + \frac{\gamma}{1 - \gamma}(\mu - r)\rho \sigma \right\} H_V + \frac{\sigma^2}{2}V^2H_{VV} = \left[ 1 - \gamma(1 - \rho^2) \right] K'(V)H,
\]

with the boundary condition \( H(V, T) = |\gamma|^{-\frac{1-\gamma(1-\rho^2)}{1-\gamma}}, \forall V \). An application of Feynman-Kac theorem then provides

\[
H(V, t) = |\gamma|^{-\frac{1-\gamma(1-\rho^2)}{1-\gamma}} \cdot E \left[ e^{-[1-\gamma(1-\rho^2)] \int_t^T K'(X_s)ds} \right| X_t = V \],
\]

with

\[
dx_s = \left\{ X_s \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log X_s) \right] + \frac{\gamma}{1 - \gamma}(\mu - r)\rho \sigma \right\} dt + \sigma X_s dZ_s,
\]

where \( Z \) is a standard Brownian motion.

2.2.3 Asymptotic solutions

We restrict our attention to the case where the investor optimizes both his/her consumption path and his/her residual wealth. Proposition 1 provides us with the optimal consumption/investment plan but requires equation (28) to be solved. It is hopeless to seek a closed form solution. That is why we propose a numerical resolution scheme appendix D that will be used in the section. Nonetheless, introducing the reduced log-volatility

\[
U := \frac{\sqrt{2}}{\sigma} (\ln V - \alpha)
\]
and performing this change of variable in equation (23), this latter can be slightly simplifies:

\[
(1-\gamma) \left( \frac{\theta^{\beta\gamma}}{\beta^{\gamma}} \right)^{-1} = K'(U)F - F_1 - \left[ \sqrt{2} \frac{\gamma (\mu - r)}{1 - \gamma} \rho \exp \left( \alpha + U \gamma \right) - \kappa \cdot U \right] F_U - F_{UU} + \gamma (1 - \rho^2) \frac{F^2}{U},
\]

(36)

with \( K'(U) := \frac{\lambda - \gamma}{1 - \gamma} - \frac{\gamma (\mu - r)^2}{2 (1 - \gamma)} \rho \exp \left( \alpha + U \gamma \right) \).

This expression is more suitable for the asymptotic analysis developed in appendix C, where we prove the following results.

**Proposition 3.** *In the limit of large volatility, i.e. \( V \to \infty \), the optimal consumption and portfolio rules given by proposition 1 can be approached by

\[
\begin{align*}
    c^*(W, t) &= K + \frac{W}{F(t)} \left( 1 - \frac{K}{W} \cdot \frac{1 - e^{-r(T-t)}}{r} \right), \\
    w^*(W, V, t) &= \frac{\mu - r}{(1 - \gamma)V^2} \left( 1 - \frac{K}{W} \cdot \frac{1 - e^{-r(T-t)}}{r} \right),
\end{align*}
\]

(37, 38)

with

\[
F(t) = \frac{1 - \gamma}{\lambda - \gamma r} + \left[ \frac{\beta^{\gamma}}{1 - \gamma} \left( \frac{1 - \theta}{\theta} \right)^{1/\gamma} - \frac{1 - \gamma}{\lambda - \gamma r} \right] e^{\frac{1 - \gamma}{\lambda - \gamma r} (T-t)}. \tag{39}
\]

In the limit of large volatilities, the volatility does not impacts the level of the investor’s consumption anymore. The asset allocation is still sensitive to the volatility even if there is no more hedging of the volatility risk. Apart from the prefactor \((1 - \gamma)^{-1}\), the agent behaves like a myopic agent irrespective of his/her risk aversion when the volatility of the risky asset is large enough.

**Proposition 4.** *In the limit of small volatility, i.e. \( V \to 0 \), the optimal consumption and portfolio rules given by proposition 1 can be approached by

\[
\begin{align*}
    c^*(W, t) &= K + \frac{W}{F(V, t)} \left( 1 - \frac{K}{W} \cdot \frac{1 - e^{-r(T-t)}}{r} \right), \\
    w^*(W, V, t) &= \left( \frac{\mu - r}{(1 - \gamma)V^2} + \rho \sigma \frac{F_V(V, t)}{F(V, t)} \right) \left( 1 - \frac{K}{W} \cdot \frac{1 - e^{-r(T-t)}}{r} \right),
\end{align*}
\]

(40, 41)

with

\[
F(V, t) = \frac{\beta^{\gamma}}{1 - \gamma} \left( \frac{1 - \theta}{\theta} \right)^{1/\gamma} e^{\frac{1 - \gamma}{\lambda - \gamma r} (T-t)} \exp \left\{ \frac{1}{2} \frac{\mu - r}{(1 - \gamma)\rho \sigma} \left[ \frac{1}{V - \frac{1}{\gamma (\mu - r)^2} \rho \sigma \left( T - t \right)} \right] \right\},
\]

(42)

\[
- \frac{\gamma - 1}{\gamma (\mu - r)^2 \rho \sigma} e^{\frac{1 - \gamma}{\lambda - \gamma r} \rho \sigma \left( V + \frac{1}{2} \frac{\mu - r}{(1 - \gamma)\rho \sigma} \right) + \left( V + \frac{1}{2} \frac{\mu - r}{(1 - \gamma)\rho \sigma} \right) \frac{1}{\gamma (\mu - r)^2 \rho \sigma} \int_V^{\infty} \frac{1}{u - \frac{1}{\gamma (\mu - r)^2 \rho \sigma} \left( T - t \right)} \left( \frac{u - \frac{1}{\gamma (\mu - r)^2 \rho \sigma} \left( T - t \right)}{u - \frac{1}{\gamma (\mu - r)^2 \rho \sigma} \left( T - t \right)} \right)^{1/2} du.
\]

3 Optimum consumption and portfolio rules in the CRRA case

In this section we develop a complete analysis under the assumption of CRRA utility function. Firstly we give the optimal policies, then we examine the investor’s saving behaviour through his optimal mean growth of wealth, and finally we present an in depth numerical analysis.
3.1 Optimal policies

We can benefit from the results obtained in the general case by noting that the CRRA utility function is a particular case of the HARA family. The CRRA utility or power function utility can be recovered from the HARA utility function by setting $\beta = (1 - \gamma)^{-1/\gamma}$ and $\eta = 0$, the utility function thus reads,

$$u(c) = \begin{cases} \frac{c^\gamma}{\gamma}, & \gamma < 1 \text{ and } \gamma \neq 0, \\ \ln c, & \gamma = 0. \end{cases}$$ (43)

The relative risk aversion coefficient is $1 - \gamma$. If $\gamma$ is close to 1, the investor is almost risk neutral, if $\gamma$ has small negative values, say $\gamma = -2$, we say he (she) is middle risk averse and if $\gamma$ takes on large negative values, say $\gamma = -20$, we say the investor is strongly risk-adverse. We focus here on the case where the economic agent maximizes his (her) consumption and the utility of terminal wealth, thus we choose $\theta = 1/2$. As a consequence of proposition 1, setting $K = 0$, we get

**Corollary 2.** In the CRRA case and in a model with a stochastic volatility defined by (3) the optimal choice verifies:

$$c^*(W, V, t) = \left(\frac{\theta}{1-\gamma}\right)^{1/\gamma} \frac{W}{F(V, t)}, \quad w^*(V, t) = \frac{\mu - r}{(1-\gamma)V^2} + \rho \sigma F(V, t),$$ (44, 45)

where $F(V, t)$ is solution to equation (23) with the same boundary condition.

We can notice that with a CRRA utility, the optimum consumption (44), at $t = T$, is such that $c^*(W, T) = \left(\frac{\theta}{1-\gamma}\right)^{1/\gamma} \cdot W$. Thus, the economic agent exhausts all his wealth at the expiry date $T$ if (and only if) $\theta = 1/2$. On the contrary, he consumes more than (resp. less) than his terminal wealth when $\theta$ is larger (resp. less) than $1/2$.

We recall here the Merton’s solution in the case of a constant volatility

$$\begin{cases} c^*_s(W, t) = \frac{W}{F(t)}, \\ w^*_s = \frac{\mu - r}{(1-\gamma)\sigma^2}, \end{cases}$$ (46, 47)

where

$$F(t) = \frac{1}{K'(\sigma_s)} + \left(\frac{1 - \theta}{\theta}\right)^{1/\gamma} \frac{1}{K'(\sigma_s)} \left(1 - e^{-K'(\sigma_s)(T-t)}\right).$$ (48)

We remark the similarities and dissimilarities in these two solutions, the structure is of the same type excepted the optimal weight $w^*_s$ which has no hedging component in the Mertonian case with constant volatility.

3.2 Optimal mean wealth growth rate

As in Merton (1969) our model allows to study the dynamic behaviour and the bequest valuation function. With a CRRA utility function and a lognormal price for the risky asset price, Merton shows that the expected growth rate of wealth

$$g(t) = \mathbb{E} \left[ \frac{dW(t)}{W(t)} \right] \frac{1}{dt}$$ (49)

9
is a decreasing function of time.

Recall Merton’s result and firstly define: \( v(t) \) as the instantaneous average propensity to consume:
\[
v(t) := c^*_s(t)/W(t) = \frac{1}{F(t)}
\]
and secondly defined by \( a^* \) the expected return on the optimal portfolio, in this constant volatility case \( a^* = r + \frac{(\mu - r)^2}{(1 - \gamma)\sigma^2} \). Then it can be shown that,
\[
g(t) = a^* - v(t) \quad (50)
\]
Depending on the initial condition the behaviour of the economic agent differs. If \( a^* < v(0) \) he (she) will disinvest, while for \( a^* > v(0) \) the individual will plan to increase his wealth until \( a^* = v(t) \), then disinvest. This behaviour is known as “hump saving.”

Now let us see what our model predicts. Replacing \( dW(t) \) and \( w(t) \) by their optimal values in (10) and (45), the expected rate of growth of wealth, now reads,
\[
g(t) = r + \frac{(\mu - r)^2}{(1 - \gamma)} E \left[ \frac{1}{V(t)^2} \right] - E \left[ \frac{1}{FV(t)} \right] + E \left[ \rho \sigma (\mu - r) \frac{FV(V(t), t)}{FV(t)} \right]. \quad (51)
\]
We remark that in the constant volatility case, we recover the Merton formula (50). The fourth term on the right hand side of the equality is new and cannot have a closed form. We have to obtain it numerically.

We can develop a little farther the computation. The random variable \( 1/V(t)^2 \) can be expressed \( e^{-2 \ln V(t)} \) which is lognormally distributed. In order to have a time independent mean we impose \( \ln V(0) = \alpha \). We deduce that in this case,
\[
g(t) = r + \frac{(\mu - r)^2}{(1 - \gamma)} e^{-2\alpha + \frac{\sigma^2}{\kappa}} + \rho \sigma (\mu - r) E \left[ \frac{FV(V(t), t)}{FV(t)} \right] - E \left[ \frac{1}{FV(t)} \right]. \quad (52)
\]
Figure 1 show that \( g(t) \) is always a decreasing function of time and depending on initial condition we have qualitatively the same pattern as in the Merton case. So once again the “hump saving” phenomenon appears. Finally let us notice that we have a closed form for \( g(T) \) because \( F(V, T) = 1 \) for all \( T \) and \( FV(V, T) = 0 \) for all \( V \):
\[
g(T) = r + \frac{(\mu - r)^2}{(1 - \gamma)} e^{-2\alpha + \frac{\sigma^2}{\kappa}} - 1, \quad (53)
\]
which is negative for all realistic values of the parameters.

### 3.3 Numerical illustration

Since no close form solution can be obtained for the investment-consumption problem presented in section 2, we now turn to a numerical analysis. The algorithm used to get numerical solutions to the problem is presented in appendix D. Thus, we consider that the risk free asset return on a yearly basis is \( r = 5\% \), the expected return of the risky asset also expressed in a yearly basis is \( \mu = 15\% \). We set \( \sigma = 1 \) and \( \kappa = 2 \) (otherwise stated a six month mean reversion). These figures are of the same order of magnitude as those reported by Masoliver and Perello (2006) which calibrates the dynamics (3-2) for the DJIA over the 20th century. The long term volatility \( \sigma_s = 30\% \), which implies (9) : \( \alpha = -1.08 \) per annum. To take into account a leverage effect we choose a negative value for the correlation between the risky asset return and the stochastic volatility: \( \rho = -0.25 \). We consider a middly risk averse investor (\( \gamma = 0 \) which gives a relative risk aversion equal to 3) and with a weak preference for the present (\( \lambda = 3\% \)). The goal of the economic agent is to maximize the aggregate utility of consumption over time with a concern about the level of his wealth at the horizon \( T = 1 \) year. The table 1 sum up this information.
<table>
<thead>
<tr>
<th>Risk free asset</th>
<th>Risk free rate</th>
<th>$r = 5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risky asset</td>
<td>Expected return</td>
<td>$\mu = 15%$</td>
</tr>
<tr>
<td>Log-volatility</td>
<td>Mean reversion</td>
<td>$\kappa = 2$</td>
</tr>
<tr>
<td></td>
<td>Long term mean</td>
<td>$\alpha = -1.08$</td>
</tr>
<tr>
<td></td>
<td>Volatility</td>
<td>$\sigma = 1$</td>
</tr>
<tr>
<td></td>
<td>Correlation</td>
<td>$\rho = -0.25$</td>
</tr>
<tr>
<td>Criterion</td>
<td>Utility + bequest</td>
<td>$\theta = 1/2$</td>
</tr>
<tr>
<td>Subjective discount rate</td>
<td>$\lambda = 3%$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Criterion</th>
<th>$\lambda = 3%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subjective discount rate</td>
<td>$\gamma = -19; -2; 0.5$</td>
</tr>
</tbody>
</table>

Table 1: Parameters of the stochastic volatility model on a yearly basis.

Our aim, here, is to present the impact of stochastic volatility on optimum policies. But before that, we firstly analyze in the lognormal case, the behaviour of optimum consumption with respect to time and volatility $\sigma_s$, and then the behaviour of the optimum weight in the risky asset with respect to this same volatility.

### 3.3.1 Impact of time and volatility in the log-normal framework

Figure 2 shows the optimum propensity to consume surface $c_s^*(t)$ as a function of the constant yearly volatility $\sigma_s$ when it varies between 0 and 100%, and time $t$ for different relative risk aversion. In the upper graph, the investor is strongly risk-adverse ($\gamma = -19$), in the middle graph he is mildly risk-adverse ($\gamma = -2$) and finally in the bottom graph he is almost risk-neutral ($\gamma = 0.5$). For each graph we have grouped the optimum consumption curves $c_s^*(t)$ obtained for the different values of the $\sigma_s$ in the range $[0, 100\%]$. These sets of curves will serve as benchmarks for the numerical study of the impact of the stochastic volatility in the next two sections.

Looking at figure 2, we observe that for high volatility values the three surfaces of consumption are increasing functions of the time and behave qualitatively similarly, in agreement with the results of corollary 1, since according to (28), $K'(\sigma_s) \rightarrow \frac{\lambda - \gamma r}{1 - \gamma}$ as $\sigma_s \rightarrow \infty$ so that the limit propensity to consume reads

$$
c_s^*(t) = \frac{\lambda - \gamma r}{1 - \gamma} \cdot \frac{1}{1 + \left[\frac{\lambda - \gamma r}{1 - \gamma} - 1\right] \cdot e^{-\frac{\lambda - \gamma r}{1 - \gamma} \cdot (T-t)}}, \quad \text{as } \sigma_s \rightarrow \infty. \quad (54)
$$

By “large volatility”, we mean $\sigma_s \gg \sqrt{\frac{\mu - r}{1 - \gamma}}$, i.e. the volatility level beyond which the economic agent stars to borrow money (the consumption profiles for such a volatility level are depicted in black in figure 2). It is actually the condition needed to satisfy $K'(\sigma_s) \approx \frac{\lambda - \gamma r}{1 - \gamma}$.

For all realistic values of the parameters $\lambda$ and $r$, as long as $\gamma$ is not too close to one, the ratio $\frac{\lambda - \gamma r}{1 - \gamma} = \lambda + \frac{\gamma (\lambda - r)}{1 - \gamma}$ remains remarkably close to $\lambda$, which explains why the consumption profiles look very similar for large values of the volatility. For example, for time $t$ equal one month, and for a 60% level of yearly volatility, the optimum propensity to consume is 53.86% if the investor is middly risk-adverse ($\gamma = -2$), while if he is strongly risk-adverse ($\gamma = -19$), the consumption level is 53.98%. Finally this level is 51.50% if he is almost risk-neutral ($\gamma = 0.5$). Besides, the first order derivative of
the consumption with respect to time, \( \partial_t c_s^* \), has the same sign as \( 1 - K'(\sigma_s) \), which goes to \( 1 - \lambda + \frac{\gamma(\lambda - r)}{1 - \gamma} \) as \( \sigma_s \) goes to infinity and then remains positive for all realistic values of the parameters \( \lambda \) and \( r \) as long as \( \gamma \) is not too close to one.

For low values of \( \sigma_s \), i.e. \( \sigma_s \ll \sqrt{\frac{\mu - r}{1 - \gamma}} \), the situation is different. The consumption is significantly impacted by the risk aversion. It decreases as the time to maturity decreases (\( \gamma = -2 \)) when volatility tends to zero while it increases for an almost risk neutral individual (\( \gamma = 0.5 \)). According to (28), \( K'(\sigma_s) \sim -\frac{\gamma(\mu - r)^2}{2(1 - \gamma)^2 \sigma_s^2} \) as \( \sigma_s \) goes to zero and therefore the first order derivative of the consumption with respect to time has the same sign as \( \gamma \) hence an increasing (resp. decreasing) propensity to consume as the time goes by when \( \gamma \) is positive (resp. negative). Surprisingly, the propensity to consume seems to remain constant for all values of \( \sigma_s \) (excepted the very smallest ones) when the individual is strongly risk-averse, as would be the case if he was myopic (\( \gamma = 0 \)). It can be rationalized by the fact that \( K'(\sigma_s) \rightarrow r \) as \( \gamma \rightarrow \infty \), i.e does not depend on \( \sigma_s \) as in the case of myopic investor for which \( K'(\sigma_s) = \lambda \) irrespective of \( \sigma_s \). Thus, as far as consumption is concerned, a strongly risk-averse investor behaves qualitatively like a myopic one, but with a subjective discount rate equal to the risk-free rate \( r \) instead of \( \lambda \).

The situation changes as far as the financial investment is concerned. Figure 3 shows the hyperbolic decay of the fraction of wealth invested in the risky asset as a function of the volatility \( \sigma_s \). Besides, this weight is also a decreasing function of the risk aversion. For a volatility level \( \sigma_s = 60\% \), a middly risk-adverse investor will invest 9.29\% of his wealth in the risky asset, while if he has a strong risk aversion he will only invest 1.39\% and a risk-neutral investor a little more than one half: 55.74\%. When the volatility becomes small enough, i.e. \( \sigma_s < \sqrt{\frac{\mu - r}{1 - \gamma}} \), the investor borrows. Everything else taken equal, the more risk averse the investor, the less he borrows. For a yearly volatility lower than 18.26\%, the fraction of wealth invested in the risky asset by the middly risk adverse investor is greater than one. The economic agent invests the totality of his wealth whose mean return is around 15\%, and takes advantage of the situation to consume more. On the contrary, when he is almost risk-neutral his consumption diminishes. In this case he borrows more: the optimal weight \( w_s^* \) is greater than one when the volatility is lower than 44.72\%.

To sum up, the risk aversion has a strong impact on the optimum rate of consumption when the level of volatility is so low that it encourages the economic agent to borrow. In such a situation, a middly risk-averse agent will consume more and will borrow to invest uniquely in the risky asset while a risk-neutral investor prefer borrowing more and decreasing his consumption. Beyond a volatility threshold such that the agent does not need to borrow anymore, the consumption rate reaches a level which increases as time goes by, irrespective of the risk aversion.

3.3.2 Stochastic volatility impact: effect of parameter \( \sigma \)

To analyze the impact of the stochastic volatility on the optimum consumption, we express the relative variation \( \frac{c_s^*(W,t) - c_s^*(W,t)}{c_s^*(W,t)} \) with respect to the relative variation: \( \frac{V - \sigma_s}{\sigma_s} \), where \( c_s^*(W,t) \) denotes the optimum consumption in a lognormal model with constant volatility \( \sigma_s \). We analyze the impact of the stochastic volatility on the optimal weight \( w^* \) in the same way: the ratio \( \frac{w^*(V,t) - w^*}{w^*} \), is expressed with respect to the relative variation of volatility where we denote by \( w^*_s \) the optimum weight obtained in the lognormal model.

We set the mean reverting parameter to six months (\( \kappa = 2 \)) and the relative risk aversion to three (\( \gamma = -2 \)). The coefficient \( \sigma \) takes on three values: \( \sigma = 0.5, 1 \) and 2. The left column of figure 4 depicts the variations of the optimal consumption with respect to the constant volatility case while the right column depicts the variations of the optimal investment in the risky asset. Both figures contain three
curves associated with three dates respectively two, six and ten months \((t = 2, t = 6 \text{ and } t = 10)\) while the expiry date is still \(T = 12\) months.

Overall, we observe that the larger the volatility, the smaller the consumption and the smaller the fraction of wealth invested in the risky asset. These results are in line with the literature. Every thing else taken equal, the impact of the uncertainty on the consumption is positive – i.e. the larger \(\sigma\), the larger the relative consumption – while it is negative on the fraction of wealth invested in the risky assets – i.e. the larger \(\sigma\), the smaller the relative investment in risky asset. To sum up, the larger the uncertainty about the future level of the volatility, the larger the consumption and the larger the relative saving in the safe asset. It means that the actual risk, i.e. the volatility \(V(t)\), and the potential risk, i.e. the uncertainty \(\sigma\), act in the same direction when we consider their impact on the investment in risky asset while they act in opposite directions when we consider their impact of consumption.

Finally, we have to mention that the impact of uncertainty on the composition of the investor’s portfolio remains moderate. On the contrary, its impact on consumption is very strong at the beginning of the period and decays at time goes by to eventually become negligible as time reaches the expiry date \(T\). This last observation is not surprising insofar as the consumption at time \(T\) is determined by a terminal condition which is independent from the volatility.

### 3.3.3 Stochastic volatility impact analyzed with \(\kappa\)

We draw the same graphs in figure 5 as in figure 4, but we reverse the role of the parameters \(\kappa\) and \(\sigma\). The uncertainty \(\sigma\) is set to 1 while the coefficient of mean reversion assumes three different values \(\kappa = 4, 2 \text{ and } 1\) which amounts to consider characteristic times of mean reversion of the volatility is respectively equal to three months, six months, and one year. Form top to bottom, the graphs in figure 5 correspond to time \(t\) equals two months, six months and ten months while the investor’s horizon is still \(T = 1\) year.

The observations about the impact of the volatility on the optimal consumption and investment in risky asset does not change: the relative consumption and the relative weight of the risky asset are decreasing functions of the volatility. As for the optimal allocation, the impact of the mean reversion is even weaker than the impact of the uncertainty (compare the right panels of figures 4 and 5). It leads us to conclude that the term of hedging \(\rho \sigma \frac{F_t(V)}{V^2}\) in (45) remains almost always negligible, so that with a very good approximation

\[
\hat{w} \simeq \frac{\mu - r}{(1 - \gamma) V^2}.
\]  

As for the consumption, the smaller \(\kappa\), the stepper the slope of the curve so that consumption is an increasing function function of \(\kappa\) for large volatilities and a decreasing function of \(\kappa\) for small ones. We can rationalize this observation if we remind that the smaller \(\kappa\) the longer the duration of the current volatility state, i.e. either high or low. Therefore, when the volatility is low, the economic agent expects the continuation of this situation, all the more so the smaller \(\kappa\). Since the consumption is a decreasing function of the volatility, the agent’s consumption is larger when both \(\kappa\) and the present volatility are smaller insofar the agent expects that the volatility will remain low for some times. Reversing the argument, we justify why the consumption is an increasing function of \(\kappa\) for large volatilities.

### Conclusion

In this article, we have suggested a new stochastic volatility model for portfolio selection and rules for optimal consumption. We have used an Ornstein Uhlenbeck process to model the logarithm of the
volatility. The interest in this modelling is firstly, that it keeps mean reversion, a well known fact for actual volatilities of financial time series, secondly this process avoids negative values for volatilities, and thirdly it can be extended to incorporate long memory effect, if for example a fractional Brownian motion is substituted to the standard Brownian motion. Thus this model can be a starting point for a model with stochastic volatility and long memory. We have developed a general methodology to find optimum policies both in the asset allocation and in consumption. This methodology rests on the classical framework, of maximizing a criterion including both utility of consumption and utility of wealth at a fixed horizon T. We have begun with HARA family, then with a CRRA utility function. We have used the dynamic programing approach, and we have solved the Hamilton Jacobi Bellman equation in two steps: firstly we have given a simplified form for this partial differential and we have then suggested an algorithm to solve it numerically. We have introduced a new concept of level of stochasticity, defined as the ratio of the spread between stochastic volatility and constant volatility divided by the constant volatility.

This analysis paves the way to study the impact of stochastic volatility on the control variables: consumption and investment. To do this we consider a middly risk averse economic agent. The stochasticity level depends crucially on two parameters $\sigma$ (the diffusion coefficient of the logarithm of volatility) and $\kappa$ (the rate of mean reversion of the log volatility). We have showed that,

- When the stochastic volatility departs from its long term level with positive values, the impact of stochasticity level controlled either by $\sigma$ or by $\kappa$, has an important impact both on consumption and investment. The impact is lower with negative values.
  - when volatility takes values greater than its long term level, the greater the stochasticity level, the lower the consumption variation and hence there is more diminution in the risky investment.
  - when volatility takes values lower than its long term level, the risk diminution of the ”stock” is partly compensated by the additional risk due to uncertainty. It is the reason why the consumption diminishes at a low pace when the volatility decreases. Besides the rise in risky investment is mitigated.

Above this threshold (???), the variation of consumption rate is a decreasing function of volatility variation.

- Contrary to $\sigma$, the influence of $\kappa$ is less obvious. It seems that when $\kappa$ increases, uncertainty decreases. Besides contrary to the uncertainty generated by $\sigma$, this emanating from $\kappa$ does not seem to add risk. Indeed,
  - although the uncertainty generated by $\kappa$ has the same influence on the consumption variation than that induced by $\sigma$,
  - it is the contrary which occurs for the variation of investment in the risky asset.

As far as the time horizon is concerned, and in general, the impact of the stochasticity of the volatility controlled by $\sigma$ and $\kappa$ on consumption diminishes while it rises on the risky part of the investment, when expiry date approaches. Near this time a middly risk averse economic agent prefers investing more in the risky asset and to reduce his consumption, privileging a kind of bequest reason. (je trouve cette conclusion bizarre et en contradiction avec ce qui est dit habituellement)

Finally, we want to mention an important modelling point concerning the leverage effect. In many models as in ours, this effect is taken into account by the correlation between the asset return and the volatility. Hence knowing the volatility at time $t$ allows to partially predict the asset future price.
Figure 1: Average growth rate of the optimal portfolio for different values $\mu = 10\%$, 20\% and 30\% of the expected return on the risky asset. The risk free rate is $r = 5\%$ and the relative risk aversion is such that $\gamma = -2$. The dashed curves refer to the lognormal case with $\sigma_s = 20\%$ while the plain curves refer to the stochastic volatility model with $\kappa = 2$, $\sigma = 1$ and a long term mean satisfying (9).
Figure 2: Optimum propensity to consume surfaces $c^*(t)$ as a function of the constant yearly volatility $\sigma_s$ when it varies between 0 and 100% and time $t$ for different relative risk aversion. The time horizon $T = 1$ year and, from top to bottom, the relative risk aversion is equal to 20 ($\gamma = -19$), 3 ($\gamma = -2$) and 0.5 ($\gamma = 0.5$). The black curves depict the consumption profiles for the values $\sigma_s = \sqrt{\frac{\mu - \gamma}{1 - \gamma}}$ at which the economic agent starts to borrow money.
Figure 3: Weight of the risky asset in the optimal portfolio as a function of $\sigma_s$ for three different values of $\gamma$. 
Figure 4: Variations of the optimal consumption (left panels) and of the optimal weight of the risky asset (right panels) as a function of the variations of the volatility relative to its long term level $\sigma_s$. The parameter of mean reversion $\kappa = 2$, the uncertainty $\sigma = 0.5$, 1 and 2 while $t = 2$, 6 and 10 months (from top to bottom).
Figure 5: Variations of the optimal consumption (left panels) and of the optimal weight of the risky asset (right panels) as a function of the variations of the volatility relative to its long term level $\sigma_s$. The uncertainty $\sigma = 1$, the parameter of mean reversion $\kappa = 1, 2$ and $4$ while $t = 2, 6$ and $10$ months (from top to bottom).
Appendix

A Proof of Proposition 1

Just recall the problem to solve

\[
0 = \max_{c(t), w(t)} \left\{ \theta e^{-\lambda t} u(c(t)) + J_t + [(\mu - r)w(t)W + rW - c(t)]J_W + \frac{\sigma^2}{2}V^2 J_{VV} \right. \\
+ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right]J_V + \frac{1}{2} w(t)^2W^2V^2J_{WW} + \rho \sigma w(t)WV^2J_{VW} \left. \right\}, \tag{56}
\]

\[
J(W, V, T) = (1 - \theta)e^{-\lambda T}B(W), \quad \forall W, V, \tag{57}
\]

where \( B(W) \) is given by (14).

First order conditions give

\[
\begin{align*}
    c^* &= u'^{-1} \left( \frac{\mu}{\theta}J_W \right), \tag{58}
    \\
    w^* &= -\frac{(\mu - r)}{W(t)V(t)^2} J_W - \frac{\rho \sigma}{W(t)} J_{VW}. \tag{59}
\end{align*}
\]

The second order conditions are

\[
\begin{align*}
    u''(c) &< 0, \tag{60}
    \\
    J_{WW} &< 0. \tag{61}
\end{align*}
\]

Substitute \( w(t) \) in equation (56) by its value in (59), we obtain the HJB equation,

\[
0 = \theta e^{-\lambda t} u(c^*) + J_t + (rW - c^*) J_W + V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] J_V \\
+ \frac{\sigma^2}{2}V^2J_{VV} - \frac{(\mu - r)^2}{2V^2} J_{WW} - \frac{\sigma^2}{2} \kappa^2 V^2 \frac{J_{VV}}{J_{WW}} - (\mu - r)\rho \sigma \frac{J_W J_{VW}}{J_{WW}}, \tag{62}
\]

with the boundary condition \( J(W, V, T) = (1 - \theta)e^{-\lambda T}B(W) \) for all \( W \) and \( V \).

When \( \gamma \neq 1 \), the HARA utility function for the consumption reads,

\[
u(c) = \frac{1 - \gamma}{\gamma} \left( \frac{\beta c}{1 - \gamma} + \eta \right)^\gamma \tag{63}
\]

From equation (58), we deduce

\[
\begin{align*}
    c^* &= \frac{1 - \gamma}{\beta} \left[ \left( \frac{e^{\lambda t} J_W}{\theta} \right)^\gamma - \eta \right] = K + \frac{1 - \gamma}{\beta} \left( \frac{e^{\lambda t} J_W}{\theta} \right)^\gamma, \tag{64}
\end{align*}
\]
where $K := \frac{(\gamma-1)\eta}{\beta}$ is the minimum subsistence level, and

$$u(c^*) = \frac{1 - \gamma}{\gamma} \left( \frac{\mu t}{\theta} J_W \right)^{\frac{-\gamma}{\beta-1}}. \tag{65}$$

Substituting $c^*$ and $u(c^*)$ given in (64) and (65) in our general PDE (62) and factoring $(J_W)^{\frac{-\gamma}{\beta-1}}$ terms, we obtain

$$0 = \frac{(1 - \gamma)^2}{\gamma} \beta^{1-\gamma} \theta e^{-\lambda t} \left( \frac{\mu t}{\theta} J_W \right)^{\frac{-\gamma}{\beta-1}} + J_t + [r W - K] J_W + V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] J_V$$

$$+ \frac{\sigma^2}{2} V^2 J_{VV} - \frac{(\mu - r)^2}{2} \frac{J_W^2}{J_{WW}} - \frac{\sigma^2}{2} \rho^2 V^2 \frac{J_{WW}^2}{J_{WW}} - (\mu - r) \rho \sigma \frac{J_W J_{WW}}{J_{WW}}. \tag{66}$$

This PDE is complicated. However, we can give it a simpler expression. Let us notice that if $\gamma = 0$ then, from equation (15), $J(W, V, t) > 0$. Conversely, when $\gamma < 0$, then $J(W, V, t) < 0$. We look for an indirect utility function $J$ that reads

$$J(W, V, t) = sgn(\gamma) e^{-\lambda t} \left[ W - K \frac{1 - e^{-r(T-t)}}{r} \right] \gamma F(V, t)^{1-\gamma}, \tag{67}$$

where $F(V, t)$ is a twice differentiable positive function. Substituting this relation in (66) obtains

$$(1 - \gamma) \left( \frac{\theta \beta^\gamma}{\gamma} \right)^{\frac{1}{1-\gamma}} = K'(V) F - F_t - \left\{ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] + \frac{\gamma}{1 - \gamma} (\mu - r) \rho \sigma \right\} F_V$$

$$- \frac{\sigma^2}{2} V^2 F_{VV} + \frac{\sigma^2}{2} (1 - \rho^2) V^2 F_V^2, \tag{68}$$

with $K'(V)$ defined by (20). The boundary condition now reads

$$F(V, T) = \left( \frac{1 - \theta}{\gamma} \right)^{\frac{1}{1-\gamma}}. \tag{69}$$

Replacing $J_W$ in the equations (64) and (65) from its value in equation (67), we get,

$$c^*(W, V, t) = K + (1 - \gamma) \left( \frac{\theta \beta^\gamma}{\gamma} \right)^{\frac{1}{1-\gamma}} \frac{W}{F(V, t)} \left( 1 - \frac{K}{W} \frac{1 - e^{-r(T-t)}}{r} \right), \tag{70}$$

$$w^*(W, V, t) = \left( \frac{\mu - r}{(1 - \gamma) V^2} + \rho \sigma \frac{F_V(V, t)}{F(V, t)} \right) \left( 1 - \frac{K}{W} \frac{1 - e^{-r(T-t)}}{r} \right), \tag{71}$$

which conclude the proof of the case $\gamma \neq 0$ in proposition 1.

As for the case $\gamma = 0$, the optimal consumption is still given by (64) and we have to solve the PDE

$$0 = -\theta e^{-\lambda t} \left[ 1 + \ln \left( \frac{\mu t}{\theta} J_W \right) \right] + J_t + [r W - K] J_W + V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] J_V$$

$$+ \frac{\sigma^2}{2} V^2 J_{VV} - \frac{(\mu - r)^2}{2 V^2} \frac{J_W^2}{J_{WW}} - \frac{\sigma^2}{2} \rho^2 V^2 \frac{J_{WW}^2}{J_{WW}} - (\mu - r) \rho \sigma \frac{J_W J_{WW}}{J_{WW}}. \tag{72}$$
with the boundary condition \( J(W, V, T) = (1 - \theta) e^{-\lambda T} \ln(W) \) for all \( W \) and \( V \). We conjecture that the solution reads

\[
J(W, V, t) = \ln \left( W - K \frac{1 - e^{-r(T-t)}}{r} \right) \cdot \left[ \frac{\theta}{\lambda} e^{-\lambda t} + (1 - \theta - \frac{\theta}{\lambda}) e^{-\lambda T} \right] + F(V, t),
\]

(73)

with \( F(V, T) = 0, \forall V \). By substitution into (72) we get

\[
V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] F_V + \frac{\sigma^2}{2} V^2 F_{VV} + \theta e^{-\lambda t} \left( r + \frac{(\mu - r)^2}{2V^2} \right) \left( \frac{1}{\lambda} + \left( \frac{1 - \theta}{\theta} - \frac{1}{\lambda} \right) e^{-\lambda (T-t)} \right)
\]

\[
= \theta e^{-\lambda t} \left[ 1 - \ln \beta + \ln \left( \frac{1}{\lambda} + \left( \frac{1 - \theta}{\theta} - \frac{1}{\lambda} \right) e^{-\lambda (T-t)} \right) \right]
\]

(74)

It is not necessary to solve this equation, since we only need \( J_W = \theta e^{-\lambda t} \left( \frac{1}{\lambda} + \left( \frac{1 - \theta}{\theta} - \frac{1}{\lambda} \right) e^{-\lambda (T-t)} \right) \) to obtain the second result stated in proposition 1.

In the lognormal case, otherwise stated the Merton constant volatility model, the partial derivatives of \( F \) with respect to \( V \) add to zero in (68) which then becomes the ordinary differential equation:

\[
(1 - \gamma) \left( \frac{\theta \beta \gamma}{|\gamma|} \right) \frac{d}{dt} = K'(\sigma_s) F - F_t,
\]

(75)

with boundary condition \( F(T) = \left( \frac{1 - \theta}{|\gamma|} \right) \frac{d}{dt} \). The solution is then given in the closed form:

\[
F(t) = \frac{1 - \gamma}{K'(\sigma_s) \left( \frac{\theta \beta \gamma}{|\gamma|} \right) \frac{d}{dt}} + \left[ \left( \frac{1 - \theta}{|\gamma|} \right) \frac{d}{dt} - \frac{1 - \gamma}{K'(\sigma_s) \left( \frac{\theta \beta \gamma}{|\gamma|} \right) \frac{d}{dt}} \right] e^{-K'(\sigma_s) (T-t)}.
\]

(76)

After a straightforward simplification, we get the result given in corollary 1.

\[\square\]

### B Proof for section 2.2.2

Here we consider the pure portfolio case. The wealth dynamics reads

\[
dW(t) = [(\mu - r)w(t)W(t) + rW(t)] dt + w(t)W(t)V(t) dZ_1(t).
\]

(77)

So the system under consideration is

\[
\begin{cases}
  dW(t) = \{(\mu - r)w(t)W(t) + rW(t)\} dt + w(t)W(t)V(t) dZ_1(t), \\
  dV(t) = \left( \frac{\sigma^2}{2} + \kappa(\alpha - \ln V(t)) \right) V(t) dt + \sigma V(t) dZ_2(t).
\end{cases}
\]

(78)

(79)

We set \( \theta = 0 \) for there is no consumption. The selection criterion becomes

\[
\max_{w(t)} \mathbb{E} \left[ e^{-\lambda t} B(W(T)) \right],
\]

(80)

with \( B(W) \) given by (14). The indirect utility function \( J(W, V, t) \) and the HJB equation respectively read

\[
J(W, V, t) = \max_{w(t)} \mathbb{E}_t \left[ e^{-\lambda t} \frac{W(T)^\gamma}{\gamma} \right],
\]

(81)

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and
\[ 0 = \max_w [\mathcal{L}J(W, t) + J_t], \tag{82} \]
where the infinitesimal differential generator is now
\[
\mathcal{L}J(W, V, t) = \{(\mu - r)w(t)W + rW\} J_W + V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] J_V
\]
\[ + \frac{1}{2}w(t)^2W^2J_{WW} + \frac{\sigma^2}{2}V^2J_{VV} + \rho\sigma w(t)WV^2J_{WV}, \tag{83} \]

The first order condition gives
\[ w^* = \frac{(\mu - r)J_W}{W \cdot V^2 J_{WW}} - \frac{\rho \sigma}{W J_{WW}}. \tag{84} \]

Substituting \( w^* \) in equation (83) by its value in (84), we obtain the HJB equation,
\[
0 = J_t + rWJ_W + V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] J_V
\]
\[ + \frac{(\mu - r)^2}{2V^2} J_{WW}^2 - \frac{\sigma^2}{2}V^2J_{WW}^2 - \frac{\sigma^2}{2}V^2J_{VV}^2 - (\mu - r)\rho \sigma \frac{J_WJ_{VV}}{J_{WW}}, \tag{85} \]
with the boundary condition \( J(W, V, T) = e^{-\lambda T} \frac{W^\gamma}{\gamma} \) for all \( W, V \). Using the same technique as in the previous appendix, we conjecture
\[ J(W, V, t) = \text{sgn}(\gamma)e^{-\lambda t} \frac{W^\gamma}{\gamma} F(V, t)^{1-\gamma}, \tag{86} \]
with \( F \geq 0 \). The optimal weight for the risky asset is then
\[ w^*(V, t) = \frac{\mu - r}{(1 - \gamma)V^2} + \frac{\rho \sigma}{F(V, t)}, \tag{87} \]
where the function \( F \) solves the PDE
\[
0 = -K'(V)F + F_t + \left\{ V \left[ \frac{\sigma^2}{2} + \kappa(\alpha - \log V) \right] + \frac{\gamma}{1 - \gamma}(\mu - r)\rho \sigma \right\} F_V
\]
\[ + \frac{\sigma^2}{2}V^2F_{VV} - \gamma \frac{\sigma^2}{2}V^2(1 - \rho^2)\frac{F_{WW}^2}{F}, \tag{88} \]
with boundary condition \( F(V, T) = \frac{1}{|\gamma|^{1-\gamma}} \) for all \( V \).

C Asymptotic results for PDE (36)

We seek a solution to the PDE (36) with the boundary condition \( F(U, T) = \left( \frac{1+\theta}{1-\theta} \right)^{1-\gamma} \) for all \( U \). Since \( F(U, T) \) is bounded, and even constant, for all \( U \), by continuity \( F \) remains bounded for all \( (U, t) \in \mathbb{R} \times [t_0, T] \) for some \( t_0 \in \mathbb{R}_+ \). As a consequence, \( F_U \) and \( F_{UU} \) goes to zero as \( U \) goes to plus or minus infinity.

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Let us first notice, for simplicity, that it is more convenient focus on the function \( F(U, t) \) which solves the PDE

\[
1 = K'(U) F - F_t - \left[ \sqrt{2} \gamma (\mu - r) \rho e^{-(\alpha + U \frac{\rho}{\gamma})} - \kappa \cdot U \right] F_U - F_{UU} + \gamma (1 - \rho^2) \frac{F_U}{F},
\]

with the boundary condition \( F(U, t) = \frac{\beta^{-\gamma}}{1-\gamma} \frac{1-\gamma}{1-\gamma} \) for all \( U \).

Let us now start with the case of large volatilities, i.e. \( U \to +\infty \). Keeping leading order terms only, equation (36) simplifies to

\[
0 = 1 - \frac{\lambda - \gamma r}{1 - \gamma} F + F_t - \kappa U \cdot F_U.
\]

Its general solution reads

\[
F(U, t) = U \frac{\gamma - \frac{1}{\lambda - \gamma r}}{\lambda - \gamma r} \cdot g(\kappa t + \ln U) + \frac{1 - \gamma}{\lambda - \gamma r},
\]

where \( g(\cdot) \) is any differentiable function. Accounting for the boundary condition and going back to the original variable \( V \), we get

\[
F(V, t) = \frac{1 - \gamma}{\lambda - \gamma r} + \left[ \beta^{\frac{-\gamma}{1 - \gamma} \left( \frac{1 - \theta}{\theta} \right)^{\frac{1 - \gamma}{1 - \gamma}} - \frac{1 - \gamma}{\lambda - \gamma r} \right] e^{-\frac{1 - \gamma}{\lambda - \gamma r}(T-t)}, \quad \text{for } V \text{ large enough.}
\]

By substitution in equations (21-22) of proposition 1 we obtain the result stated in proposition 3.

Considering now the case of small volatilities, i.e. \( U \to -\infty \), when \( \gamma \neq 0 \) and keeping leading order terms only, equation (36) becomes

\[
1 = \left[ \frac{\lambda - \gamma r}{1 - \gamma} \frac{\gamma (\mu - r)^2}{2 (1 - \gamma)^2} e^{-(\alpha + U \frac{\rho}{\gamma} \sqrt{2})} \right] F(U, t) - F_t(U, t) - \sqrt{2} \gamma (\mu - r) e^{-(\alpha + U \frac{\rho}{\gamma})} F_U(U, t).
\]

Its general solution reads

\[
F(U, t) = e^{\gamma (\mu - r) \rho \sigma} e^{\frac{\alpha + U \frac{\rho}{\gamma} \sqrt{2}}{2 (1 - \gamma)^2} \rho \sigma} e^{-\frac{\alpha + U \frac{\rho}{\gamma}}{2 (1 - \gamma)^2}} \left[ g\left( t - \frac{1 - \gamma}{\gamma (\mu - r) \rho \sigma} : e^{\alpha + U \frac{\rho}{\gamma} \sqrt{2}} \right) + G(U) \right],
\]

where \( g(\cdot) \) is any differentiable function and

\[
G(U) = \frac{\gamma - 1}{\gamma (\mu - r) \rho \sigma} \int_0^e \frac{e^{\frac{\alpha + U \frac{\rho}{\gamma} \sqrt{2}}{2 (1 - \gamma)^2} \rho \sigma} - u - \frac{1}{2 (1 - \gamma)^2}}{e^{-\gamma (\mu - r) \rho \sigma} - u + \frac{1}{2 (1 - \gamma)^2}} du.
\]

Thus, accounting for the boundary condition, and expressed in terms of the original variable \( V \), we obtain

\[
F(V, t) = \beta^{\frac{-\gamma}{1 - \gamma} \left( \frac{1 - \theta}{\theta} \right)^{\frac{1 - \gamma}{1 - \gamma}} \exp \left( \frac{1}{2 (1 - \gamma)^2} \rho \sigma \right) \left[ \frac{1}{V} - \frac{1}{\gamma (\mu - r) \rho \sigma} \int_0^e \frac{e^{\frac{\alpha + U \frac{\rho}{\gamma} \sqrt{2}}{2 (1 - \gamma)^2} \rho \sigma} - u - \frac{1}{2 (1 - \gamma)^2}}{e^{-\gamma (\mu - r) \rho \sigma} - u + \frac{1}{2 (1 - \gamma)^2}} du \right] \]

\[
- \frac{\gamma - 1}{\gamma (\mu - r) \rho \sigma} e^{\frac{\lambda - \gamma r}{\gamma (\mu - r) \rho \sigma} - \frac{1}{2 (1 - \gamma)^2} \rho \sigma} \cdot \frac{1}{V} \int_0^e \frac{e^{\frac{\alpha + U \frac{\rho}{\gamma} \sqrt{2}}{2 (1 - \gamma)^2} \rho \sigma} - u - \frac{1}{2 (1 - \gamma)^2}}{e^{-\gamma (\mu - r) \rho \sigma} - u + \frac{1}{2 (1 - \gamma)^2}} du \]

\[
\approx \beta^{\frac{-\gamma}{1 - \gamma} \left( \frac{1 - \theta}{\theta} \right)^{\frac{1 - \gamma}{1 - \gamma}} \exp \left( \frac{1}{2 (1 - \gamma)^2} \rho \sigma \right) \left[ \frac{1}{V} - \frac{1}{\gamma (\mu - r) \rho \sigma} \int_0^e \frac{e^{\frac{\alpha + U \frac{\rho}{\gamma} \sqrt{2}}{2 (1 - \gamma)^2} \rho \sigma} - u - \frac{1}{2 (1 - \gamma)^2}}{e^{-\gamma (\mu - r) \rho \sigma} - u + \frac{1}{2 (1 - \gamma)^2}} du \right] \]

\[
- \frac{\gamma - 1}{\gamma (\mu - r) \rho \sigma} \frac{1}{V} \int_0^e \frac{e^{\frac{\alpha + U \frac{\rho}{\gamma} \sqrt{2}}{2 (1 - \gamma)^2} \rho \sigma} - u - \frac{1}{2 (1 - \gamma)^2}}{e^{-\gamma (\mu - r) \rho \sigma} - u + \frac{1}{2 (1 - \gamma)^2}} du \]
in the limit of small volatilities. By substitution in equations (21-22) of proposition 1 we obtain the result stated in proposition 4.

D Numerical solution to the PDE

In the dynamic programming approach of portfolio selection in continuous time the main difficulty is to solve the Hamilton Jacobi Bellman equation. In our model we have to solve the non linear PDE (23). We cannot hope to solve it in a closed form, so we choose a numerical solution and suggest a new algorithm. Performing the change of variable (35), we obtain (36) which can be numerically solved by a recursive scheme. It turns out that this scheme is much more stable when we consider the variable $U$ instead of $V$, hence our choice to resort to (36) instead of (23). We rewrite this equation as

$$F_t(U,t) = -1 + A \cdot F(U,t) + B \cdot F_U(U,t) - F_{UU}(U,t) + \gamma (1 - \rho^2) \frac{F_U(U,t)^2}{F(U,t)},$$

for all $U \in \mathbb{R}$ and $\forall t \in [0, T]$, where $A$ and $B$ are defined by

$$A := \frac{\lambda - \gamma r}{1 - \gamma} + \frac{\gamma (\mu - r)^2}{2(1 - \gamma)^2} e^{-2(\alpha + U \frac{\sigma}{\sqrt{2}})},$$

$$B := \kappa \cdot U - \sqrt{2} \gamma (\mu - r) \rho e^{-\left(\alpha + U \frac{\sigma}{\sqrt{2}}\right)}.$$

The algorithm we propose is the following:

1. Make the time discretization : $t = [0, \tau, 2\tau, \cdots, T - \tau, T]$, where $\tau := T/N$,

2. Start with the terminal condition $F(U,T)$ whose value is known,

   (a) Deduce $F_U(U,T) = 0$ and $F_{UU}(U,T) = 0$,

   (b) Obtain $F_t(U,T) = A - 1$ thanks to (99),

   (c) Obtain $F(U,T - \tau) \approx F(U,T) - \tau F_t(U,T)$,

3. Then, for $k = 1 \cdots N - 1$,

   (a) Use $F(U,T - k\tau)$ which has just been computed,

   (b) Deduce $F_U(U,T - k\tau)$ et $F_{UU}(U,T - k\tau)$ by a finite difference method,

   (c) Compute $F_t(U,T - k\tau)$ thanks to (99),

   (d) Finally obtain $F(U,T - (k + 1)\tau) \approx F(U,T - k\tau) - \tau F_t(U,T - k\tau)$,

This algorithm provides any desired value of $F(U,t)$, and then of $F(V,t)$, for any $t$ in $\{0, \tau, 2\tau, \cdots, T\}$. As a byproduct, we obtain $F_V(V,t)$ which is necessary to get $w^*(V,t)$ by (45).
References


