Credit Risk valuation with rating transitions and partial information

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Abstract

This work intend to shed some light on a new use of Phase-type distributions in credit risk, taking into account different flows of information without huge numerical calculations. We consider credit migration models with partial information and studies the influence of a deficit of information on prices of credit linked securities. The transitions through the various credit classes are modeled via a homogeneous continuous time Markov chain but they are not directly observable by investors in the secondary market. We first consider the case of one bond issuer and study three settings of partial information. In a first model, information about ratings arrives at predetermined dates with delay periods. In a second model, information arrives randomly according to an exogenous Poisson process, whereas in a third model, information arrives randomly according to an endogenous rule (transitions are observed only when they lead the Markov chain to a class with a lower credit rating). We infer in the three settings bonds and options prices, and we provide an explicit description of the dynamics of bond prices under real and pricing measures. We also consider the case of two bond issuers and we analyze the cross effects of deficit of information and contagion on bonds prices and correlation of default times. We conclude the paper by numerical illustrations.

Keywords. Credit migrations models; Partial and delayed information; Corporate bonds; Credit derivatives; Phase-type distributions.

1 Introduction

There are two main types of models that attempt to describe default processes in the credit risk literature: structural and reduced form models. Structural models use the evolution of firms structural variables, such as asset and debt values, to determine the time of default. In these models, it is usually characterized as the first hitting time of the firm’s asset value (modeled as a diffusion) to a given boundary determined by the firm’s liabilities. This time of default is usually a predictable stopping time. In contrast, reduced form models do not consider the relation between default and firm value in an explicit manner and the credit events are specified in terms of some exogenous given jump processes. It is possible to distinguish between the reduced-form models that are only concerned with the modeling of the time of default, called the intensity-based models, and those with migrations between credit rating classes, called the credit migration models. In this approach the time of default is a totally inaccessible stopping time. Reviews of credit risk models can be found in many books, including among others Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Schönbucher (2003), and Lando (2004).

A quick look at the above description could lead one to conclude that structural and reduced-form models are competing paradigms. However, an intrinsic connection between these two approaches has been pointed out in the last decade by several authors. Actually it is possible to
show that structural models can be viewed as reduced-form models by secondary markets that have only incomplete information. In the seminal paper, Duffie and Lando (2001), the firm asset value is modeled as a continuous-time process but the market has only at discrete point times a noisy accounting report of this value and knows the default history of the firm. In this setting, Duffie and Lando establish that the default time admits an intensity that is proportional to the derivative of the conditional density of the firm’s value at the default barrier. Structural models with incomplete information have also been studying by Kusuoka (1999), Nakagawa (2001), Jarrow and Protter (2004), Coculescu et al. (2008), Frey and Schmidt (2009), Guo et al. (2009) and Frey and Dan Lu (2012) among others.

Reduced form credit risk models with incomplete information have been less studied. Schönbucher (2004), Collin-Dufresne et al. (2003) and Duffie et al. (2006) considered models where default intensities are driven by an unobservable factor process and the information of the secondary market is given by the default history of the portfolio of obligors, augmented by some economic covariates. Given information about the factor process, the default times of obligors are assumed to be conditionally independent, doubly stochastic random times. More recently Frey and Runggaldier (2010) and Frey and Schmidt (2012) have considered general reduced form pricing models where default intensities are driven by some factor process which are not directly observable by investors in secondary markets. Their information set only consists of the default history and of observation of noisy prices for traded credit securities.

Our paper considers credit migration models with partial information and studies the influence of a deficit of information on prices of credit linked securities. It differs from the previous contributions in several directions: First, instead of using a single credit rating intensity based model, we rather work in a multiple credit ratings framework where we can look in greater details at the changes in credit quality that may lead to default. The transitions through the various credit classes are modeled via a homogeneous continuous time Markov chains. Second, with this framework, it is possible by using the properties of Phase-type distributions to derive some explicit pricing formulas for the prices of risky bonds and options on these risky bonds. The influence of a deficit of information is easily quantified without requiring intensive simulation of the credit rating. Third, with regard to information, we consider a framework of discretely delayed filtrations as introduced in Guo et al. (2009), i.e. no new information flow in between two consecutive observation times as long as the issuer defaults. Finally, we propose in this framework, a simple contagion models to price credit derivatives on two firms. Remark that models proposed in this work could be adapted to model queuing systems with incomplete information, such these studied by Xiaoqiang et al. (2009).

The remainder of the paper is organized as follows. In Section 2, we consider several models with one issuer under three discontinued flows of information. First, we assume that investors observe the issuer’s rating only at discrete and deterministic points with a fixed delay. Second, we assume that information arrives randomly, but exogenously, according to a Poisson process. Third, we assume that the ratings are observed at deterministic points or when transitions lead the Markov chain to a class with a lower credit rating than the last rating known. In this way information also arrives randomly, but according to an endogenous rule. We infer in the three settings bonds and options prices and we provide an explicit description of the dynamics of bond prices under real and pricing measures. In Section 3, we consider a contagion model with two issuers and analyze the cross effects of a deficit of information and contagion on bonds prices and correlation of default times. We discuss the models of the two previous sections conclude the paper by numerical illustrations in Section 4.

2 Several models for one issuer’s bond with three types of partial information

In this section, we propose three models to price corporate or sovereign bonds under a discontinued flow of information. The firm or the country that issues the bond is marked in a rating system
counting \( L \) levels. The levels 1 and \( L - 1 \) correspond respectively to the highest and to the lowest credit qualities. When the issuer reaches the level \( L \), it goes into bankruptcy and the bond pays \( R \), the recovery rate, at maturity \( T \).

Information about the current rating is carried by \( (X_t) \), a continuous-time, time-homogeneous Markov chain defined on a probability space \((\Omega, \mathcal{F}, P)\) where \( P \) denotes the real probability measure. The Markov chain takes its values in a finite state space \( E = \{e_1, e_2, \ldots, e_L\} \) where \( e_i \) is the \( i \)-th vector of the standard basis of \( \mathbb{R}^L \) (the \( j \)-th component of \( e_i \) is the Kronecker delta, \( \delta_{i,j} \), for each \( i = 1, \ldots, L \)). The dynamics of the chain \((X_t)\) is specified by the \( L \times L \) matrix of intensities of transition, \( \Gamma = (\gamma_{i,j})_{i,j=1,2,\ldots,L} \). The elements of \( \Gamma \) satisfy the following conditions

\[
\gamma_{i,j} \geq 0 \quad \forall i \neq j \quad \sum_{j=1}^{L} \gamma_{i,j} = 0 \quad i = 1, \ldots, L.
\]

\( \Gamma \) is related to the probability of transition from rating \( i \) to \( j \), on a period of time \( \Delta t \), as follows

\[
p_{i,j}(t, t + \Delta t) = p_{i,j}(\Delta t) = e_i^T \exp(\Gamma \Delta t) e_j,
\]

where \( ^T \) denotes the transpose operator and \( \exp(\Gamma \Delta t) \) is the exponential of the matrix \( \Gamma \Delta t \). Given that the issuer defaults when it reaches the rating \( L \), the matrix of intensities can be rewritten in the following block form

\[
\Gamma = \begin{pmatrix} \Lambda & \varsigma \\ 0 & 0 \end{pmatrix}
\]

where \( \varsigma = (\varsigma_1, \ldots, \varsigma_{L-1})^T \) is a \( L - 1 \) dimensional vector with non-negative components and \( \Lambda \) is a \((L-1) \times (L-1)\) matrix such that \( \varsigma = -\Lambda 1_{L-1} \) (with \( 1_{L-1} = (1, \ldots, 1)^T \)). We introduce the matrix \( D \) defined as

\[
D = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix} = \begin{pmatrix} Id_{L-1} & 0_{L-1} \end{pmatrix}
\]

where \( Id_{L-1} \) is the identity matrix of dimension \( L - 1 \). The product \( DX_t \) is then the vector of the \( L - 1 \) first elements of \( X_t \). We assume that the state \( e_L \) is the only recurrent state of the chain. In this setting, the time of default, \( \tau = \inf \{ t > 0 : X_t = e_L \} \), has a Phase-type distribution and the probability of survival is given by (see e.g. Rolski et al. (2009))

\[
P(\tau > t|\mathcal{F}_0) = X_0^T D^T \exp(\Lambda t) 1_{L-1}.
\]

Let us introduce the following processes

\[
H^i_t = 1_{\{X_t=e_i\}}, \quad H^{i,j}_t = \sum_{0 < u \leq t} 1_{\{X_u=e_i\}} 1_{\{X_u=e_j\}} \quad \text{and} \quad M^{i,j}_t = H^{i,j}_t - \int_0^{\tau} \gamma_{i,j} H_t^i du \quad \text{for } i \neq j.
\]

It is well-known that \( M^{i,j}_t \) is a \( \mathcal{F} \)-martingale under the measure \( P \) (see e.g. Lemma 11.2.3 in Bielecki and Rutkowski (2002)).

We may now identify the dynamics of the rating process \((X_t)\) under the pricing measure \( Q \). To achieve this goal, we assume that \((X_t)\) is also a time-homogeneous Markov chain defined by a \( L \times L \) matrix of intensities of transition, denoted by \( A = (a_{i,j})_{i,j=1,2,\ldots,L} \), that satisfies the same types of conditions as for \( \Gamma \). Since \( Q \) is an equivalent measure to \( \tilde{P} \), the matrix \( A \) can be rewritten in the following block form

\[
A = \begin{pmatrix} B & b \\ 0 & 0 \end{pmatrix}
\]

where \( b = -B 1_{L-1} \). Let us now introduce the “costs” of transition \( \kappa_{i,j} \) defined as

\[
a_{i,j} = (1 + \kappa_{i,j}) \lambda_{i,j}, \quad \text{for } i \neq j,
\]

(2.1)
and assume that $\kappa_{i,j} > -1$. By virtue of Proposition 11.2.3 in Bielecki and Rutkowski (2002), the probability measure $Q$ is characterized by the Radon Nikodym density $\eta_t$ given by

$$
\eta_t = 1 + \int_{(0,t]} \eta_u - \sum_{i,j=1,i\neq j}^L \kappa_{i,j} dM_{i,j}^u
$$

and is such that the generators of $(X_t)$ under $P$ and $Q$ are respectively $\Gamma$ and $A$. The time of default has also a Phase-type distribution under the pricing measure $Q$ and the probability of survival is now given by

$$
P^Q(\tau > t | \mathcal{F}_0) = X_0^T D_\tau \exp(B t) 1_{L-1}.
$$

The use of continuous-time, time-homogeneous Markov-chain to model transitions between credit rating has been first proposed by Jarrow et al. (1997). Here we further assume that the dynamics of $(X_t)$ under $Q$ is also time-homogeneous to exploit the properties of Phase-type distributions and to discuss the influence of a deficit of information. However note that it is possible to weaken this assumption and generalize our result by considering extensions to take into account memory and stochastic transitions as in Arvanitis et al. (1999).

Since their introduction by Neuts in 1975 and their generalization in a multivariate setting by Assaf et al. in 1983, Phase-type distributions have been used in a wide range of stochastic modelling applications in areas as diverse as telecommunications, teletraffic modelling, queuing theory, reliability, ruin theory, healthcare systems modelling (see e.g. Fackrell (2003) (2008), Pérez-Ocón (2011)). However, their use in modelling financial risks is still recent: to the best of our knowledge, multivariate Phase-type distribution have been considered to model default contagion (Herbertsson and Rootzén (2008), Herbertsson (2011)) and to value synthetic CDO tranche spreads, index CDS spreads, $k^{th}$-to-default swap spreads and tranchelets (Herbertsson (2008)).

### 2.1 A first model for the defaultable bond with information at fixed dates

In practice, the credit quality of issuers is only assessed at discrete times, when credit rating agencies assign and publish credit ratings for issuers. No information on the credit worthiness is available between these times as long as the issuer defaults (which is supposed to be instantaneously observed). In this section, we assume that times of credit rating publications, $0 = t_{0}^{(p)} \leq t_{1}^{(p)} \leq \ldots \leq t_{n}^{(p)} = T$, are non random and that they integrate a delay, $\delta$, due to the time needed for verification of the issuer’s credit-worthiness. We denote by $t_{i} = t_{i}^{(p)} - \delta$, $i \geq 1$, the times when the credit rating ratings are really assessed and let $t_{0} = t_{0}^{(p)} = 0$. As pointed out in Guo et al. (2009), this seems to be a realistic assumption as long as it can be assumed that investors observe ratings when credit rating agencies provide their annual or quarterly reports.

If $H_t$ is the indicator variable $1_{\{\tau > t\}}$, equal to one if the issuer is still solvent, the information available to the secondary market is represented by the filtration $\{\mathcal{G}_t, t \leq T\}$ where the sigma algebra $\mathcal{G}_t$ at time $t \in [t_{i}^{(p)}, t_{i+1}^{(p)+1]}$ is given by $\mathcal{G}_t = \sigma \{X_{t_{0}}, \ldots, X_{t_{i}}, H_u, u \leq t\}$.

As our purpose is to illustrate the impact of the lack of information on bond prices, we assume that the risk free rate, noted $r$, is constant. This assumption can easily be released without introducing major modifications in the main results of the paper.

Let us denote by $P(t,T)$ the price at time $t \leq T$ of a zero-coupon bond, that delivers one monetary unit at maturity $T$. The price of the defaultable bond is equal to the expected discounted payoff of the bond, conditionally to the available information

$$
P(t, T) = e^{-r(T-t)} \mathbb{E}^Q (R + (1-R) H_T | \mathcal{G}_t).
$$

(2.2)

The following proposition gives an explicit expression for $P(t,T)$.
Proposition 2.1. Suppose that \( t \in [\hat{t}_i^{(p)}, \hat{t}_{i+1}^{(p)}) \) for some \( i = 0, \ldots, n - 1 \), then
\[
P(t, T) = Re^{-r(T-t)} + (1 - R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t, T)H_t,
\]
where
\[
\gamma(X_{t_i}, t_i, t, T) = P Q (\tau > T | X_{t_i}, t < \tau) = \frac{X_{t_i}^T D^T \exp (B (T - t_i)) 1_{L-1}}{X_{t_i}^T D^T \exp (B (t - t_i)) 1_{L-1}}. \tag{2.3}
\]

All proofs are in section 6. Note that, if the issuer is still solvent at time \( t = \hat{t}_i^{(p)} - \), we observe a jump in the price of amplitude
\[
\Delta P(t, T) = (1 - R)e^{-r(T-t)} \left( \gamma(X_{t_i}, t_i, \hat{t}_i^{(p)}, T) - \gamma(X_{t_{i-1}}, t_{i-1}, \hat{t}_i^{(p)}, T) \right).
\]

Remark 2.2. In case of default, the bond pays \( R \) at maturity \( T \) whatever the time of default. In our time-homogeneous Markov framework, it is possible to analyze the effect on the price of the immediate delivery of the recovered part of the nominal. Assume that, when the issuer reaches the level \( L \), it goes into bankruptcy and the bond pays \( R \) at time \( \tau \). For \( t \in [\hat{t}_i^{(p)}, \hat{t}_{i+1}^{(p)}) \) and \( i = 0, \ldots, n - 1 \), the price of the zero coupon bond is given by (see Section 6.2 for a proof)
\[
P(t, T) = Re^{r(t-\tau)} (1 - H_t) + H_t R \sum_{j=1}^{L-1} \frac{\eta_{j}^T D^T (r I_{D_j} - B)^{-1} b X_{t_i}^T \exp (A(t - t_i)) e_j}{X_{t_i}^T D^T \exp (B(t - t_i)) 1_{L-1}} \tag{2.4}
\]
\[
+ e^{-r(T-t)} H_t \frac{X_{t_i}^T D^T \exp (B (T - t_i)) 1_{L-1}}{X_{t_i}^T D^T \exp (B (t - t_i)) 1_{L-1}} - R e^{-r(T-t)} H_t \sum_{j=1}^{L-1} \frac{\eta_{j}^T D^T (r I_{D_j} - B)^{-1} b X_{t_i}^T \exp (A(T - t_i)) e_j}{X_{t_i}^T D^T \exp (B(t - t_i)) 1_{L-1}}.
\]

We now derive the dynamics of the zero coupon bond under the reduced filtration by using the innovations approach.

Proposition 2.3. Suppose that \( t \in (\hat{t}_i^{(p)}, \hat{t}_{i+1}^{(p)}) \) for some \( i = 0, \ldots, n - 1 \). The dynamics of the bond price under \( Q \) and with partial information is given by
\[
dP(t, T) = r P(t, T)dt - (1 - R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t, T) d\tilde{M}_t^0,
\]
where \( (\tilde{M}_t^0) \) is a \( (\mathcal{G}, Q) \)-martingale.

Remark 2.4. It is also possible to give the dynamics of the bond price under \( P \) and with partial information by using Equation (2.1). Suppose that \( t \in (\hat{t}_i^{(p)}, \hat{t}_{i+1}^{(p)}) \) for some \( i = 0, \ldots, n - 1 \)
\[
dP(t, T) = r P(t, T)dt + (1 - R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t, T) \times
\]
\[
\sum_{j=1}^{L-1} \kappa_{j} a_{j} \rho_{j} \rho \frac{X_{t_i}^T \exp (\Gamma (t - t_i)) e_j}{X_{t_i}^T D^T \exp (A(t - t_i)) 1_{L-1}} dt
\]
\[
- (1 - R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t, T) d\tilde{M}_t^{0,P},
\]
where \( (\tilde{M}_t^{0,P}) \) is a \( (\mathcal{G}, P) \)-martingale. Moreover, at \( t = \hat{t}_i^{(p)} - \), \( \mathbb{E}^Q (\Delta P(t, T) | \mathcal{G}_t) = 0 \).

We finally focus on the calculation of the price of an call option on this zero coupon bond with maturity \( S \leq T \) and strike price \( K \):
\[
C(t, S, K, T) = \mathbb{E}^Q \left( \left( e^{-r(S-t)} (P(S, T) - K)^+ \right) | \mathcal{G}_t \right).
\]
The interest rate being deterministic, the only factor of risk is the default risk.
Proposition 2.5. Suppose that \( t \in [t_i^{(p)}, t_{i+1}^{(p)}) \) for some \( i = 0, \ldots, n - 1 \). The price of a call option of exercise date \( S \in [t_S^{(p)}, t_{S+1}^{(p)}) \) \((t_i^{(p)} \neq t_S^{(p)})\) and of strike price \( K \) is given by
\[
C(t, S, K, T) = \sum_{j=1}^{L-1} \delta(X_{t_i}, e_j, t_i, t, S) e^{-r(S-t)} \left( e^{-r(T-S)} (R + (1 - R)\gamma(e_j, t_S, S, T)) - K \right) + H_t
\]
\[
+ (1 - \gamma(X_{t_i}, t_i, t, S) H_t) e^{-r(S-t)} \left( e^{-r(T-S)} R - K \right) +
\]
where \( \gamma(\cdot) \) is defined by Equation (2.3) and
\[
\delta(X_{t_i}, e_j, t_i, t, S) = P^Q(X_{t_S} = e_j \mid X_{t_i}, t < \tau) P^Q(\tau > S \mid X_{t_S} = e_j, t < \tau)
\]
\[
= \frac{X_{t_i}^T \exp(A(t_S-t_i)) e_j}{X_{t_i}^T D^T \exp(B(t-t_i)) 1_{L-1}} e_j D^T \exp(B(S-t_S)) 1_{L-1}.
\]

2.2 A second model for the defaultable bond with information at exogenous random times

In the previous model, the credit quality was available at discrete and deterministic times. We now assume that the rating is disclosed at random times till bankruptcy according to a homogeneous Poisson process (if the company or country is in default, no more information is disclosed). The random times of (possible) rating disclosures are denoted by \( Z_k, k \geq 0 \). The intervals of time between two successive publication dates of rating, \( U_k = Z_k - Z_{k-1} \), are assumed to be independent and identically distributed as exponential random variables with mean \( \nu^{-1} \) under the real probability \( P \). Moreover they are independent of the Markov chain \( (X_t) \). The number of possible rating disclosures at time \( t \) is denoted by \( N_t \) and the number of real rating disclosures at time \( t \) is denoted by \( N^*_t \). Its distribution under \( P \) is given by
\[
P(N^*_t = n \mid F_0) = \int_0^t \frac{(\nu u)^n}{n!} e^{-\nu u} X_0^T D^T \exp(\Lambda u) \zeta du + \frac{(\nu t)^n}{n!} e^{-\nu t} X_0^T D^T \exp(\Lambda t) 1_{L-1}.
\]
We assume that \( (N_t) \) is a homogeneous Poisson process of intensity \( \lambda \) under the pricing measure \( Q \). As in the first model, we introduce the cost of information, \( \kappa_P \), such that
\[
\lambda = (1 + \kappa_P)\nu.
\]
By virtue of Proposition 11.2.3 in Bielecki and Rutkowski (2002), the probability measure \( Q \) is defined by the product of Radon Nikodym densities \( \eta_t \) and \( \zeta_t \) given respectively by
\[
\eta_t = 1 + \int_{(0,t]} \eta_{u-} \kappa_{i,j} dM^i_{u,j}, \quad \zeta_t = 1 + \int_{(0,t]} \zeta_{u-} \kappa_P d (N_u - \nu u),
\]
and is such that generators of \( (X_t) \) under \( P \) and \( Q \) are \( \Gamma \) and \( A \) respectively and that the frequencies of information arrivals under \( P \) and \( Q \) are respectively \( \nu \) and \( \lambda \).

The information available to the market is represented by the filtration \( \{\mathcal{G}_t, t \leq T\} \) that carries now information about the disclosure dates, the rating at these times and the occurrence of default. The sigma algebra \( \mathcal{G}_t \) is then given by
\[
\mathcal{G}_t = \sigma\left\{N^*_t, Z_0, Z_1, \ldots, Z_{N^*_t}, X_{Z_0}, \ldots, X_{Z_{N^*_t}}, H_u, u \leq t\right\}.
\]
The price of the defaultable bond is equal to the expected discounted payoff of the bond, conditionally to the available information
\[
P(t, T) = e^{-r(T-t)} E^Q(R + (1 - R) H_T \mid \mathcal{G}_t) \tag{2.5}
\]
and has the same expression as in the previous model.
Proposition 2.6. The price at time $t$ of a zero coupon of maturity $T$ is

$$P(t, T) = Re^{-r(T-t)} + (1 - R)e^{-r(T-t)}\gamma(X_{Z_{N_{t}^-}}, Z_{N_{t}^-}, t, T)H_t. $$

The proof is similar to the proof of Proposition 2.1 and is left for the reader who may easily supply the details. We now focus on the dynamics of the price of the defaultable bond.

Proposition 2.7. The dynamics at time $t$ of the bond price under $Q$ and with partial information is given by

$$dP(t, T) = rP(t, T)dt - (1 - R)e^{-r(T-t)}\gamma(X_{Z_{N_{t}^-}}, Z_{N_{t}^-}, t, T) dM_t^0 + (1 - R)e^{-r(T-t)} \left( \gamma(X_t, t, t, T) - \gamma(X_{Z_{N_{t}^-}}, Z_{N_{t}^-}, t, T) \right) dN_t^r$$

where $M_t^0$ and is a $(\mathcal{G}, Q)$-martingale.

Remark 2.8. It is also possible to give the dynamics of the bond price under $P$ and with partial information

$$dP(t, T) = rP(t, T)dt - (1 - R)e^{-r(T-t)}\gamma(X_{Z_{N_{t}^-}}, Z_{N_{t}^-}, t, T) dM_t^{0,P} + (1 - R)e^{-r(T-t)}\gamma(X_{Z_{N_{t}^-}}, Z_{N_{t}^-}, t, T) \sum_{j=1}^{L-1} \kappa_j, \alpha_j, \bar{L}H_t \frac{X_t^j}{X_t} \exp(\Gamma(t - t_i)) e_j dt$$

$$+ (1 - R)e^{-r(T-t)} \left( \gamma(X_t, t, t, T) - \gamma(X_{Z_{N_{t}^-}}, Z_{N_{t}^-}, t, T) \right) dN_t^r,$$

where $(M_t^{0,P})$ is a $(\mathcal{G}, P)$-martingale.

Corollary 2.9. The expected return of the bond is equal to the risk free rate

$$\mathbb{E}^Q (dP(t, T) | \mathcal{G}_t) = rP(t, T) dt.$$  

We now focus on the calculation of the price of an option on this zero coupon bond,

$$C(t, S, K, T) = \mathbb{E}^Q \left( (e^{-r(S-t)} (P(S, T) - K)_+) | \mathcal{G}_t \right).$$

The interest rate being deterministic, the only factor of risk is the default risk.

Proposition 2.10. Let us denote by $z = Z_{N_{t}^-}$, the date of the last rating disclosure before time $t$. The price of a call option of exercise date $S$ and of strike price $K$ is given by the following expression

$$C(t, S, K, T) = \mathbb{H}_t \sum_{j=1}^{L-1} \int_t^S \delta(X_z, e_j, z, t, v, S) e^{-r(S-t)} \left( e^{-r(T-S)} (R + (1 - R)\gamma(e_j, v, S, T) - K) \right) + f_V(v)dv$$

$$+ (1 - \gamma(X_z, z, t, S)H_t) e^{-r(S-t)} \left( (e^{-r(T-S)} R - K) \right)$$

where $f_V(v)$ is the density of the time of the last rating disclosure, $V = Z_{N_S}$, in the interval of time $[t, S]$ given that $\tau > S$,

$$f_V(v) = e^{-\lambda(S-t)} \left( \delta(v - t) + \lambda e^{-\lambda(t-v)} \right).$$

with $\delta(.)$ the Dirac function.
2.3 A third model for the defaultable bond with information at endogenous dates

In this last model, we assume that the rating is disclosed at discrete and deterministic times
$0 = t^{(p)}_0 \leq t^{(p)}_1 \leq \ldots \leq t^{(p)}_n = T$, but also at random and endogenous times, when transitions lead
the Markov chain to a class with a lower credit rating than the last rating known. This is a more
realistic assumption since credit rating agencies not only provide ratings for their annual reports,
but also when the credit worthiness of the firm or of the country becomes worse than the last known
rating could let to predict. This assumption introduces a kind of asymmetry in the information
arrival process: an improvement of credit worthiness is not immediately disclosed, contrary to
worsening. We denote by $0 = Z_0 \leq Z_1 \leq \ldots$ the times of rating disclosures (deterministic times
as well as at random times) and by $N^{\tau}_t$ the number of rating disclosures before time $t$. The
information available to the market is represented by the filtration $\{G_t, t \leq T\}$ where $G_t$ is then
given by

$$G_t = \sigma \left\{ N^\tau_t, t_0, Z_1, ..., Z_{N^\tau_t}, X_{Z_0}, ..., X_{Z_{N^\tau_t}}, H_u, u \leq t \right\}.$$ 

The rating of the issuer at time $t$ is denoted by $R_t$ and is equal to the scalar product of $X_t$ and
of the $L$ dimensional vector $(1, 2, ..., L)^T$. The price of the defaultable bond has not the same
expression as in the previous models because the length between the last rating disclosure is now
informative. Let us introduce three families of matrices: for rating $R = 1, ..., L - 1$

$$B^{\tau R} = (b_{ij})_{1 \leq i \leq R, 1 \leq j \leq R}, \quad Id^{\tau R} = (1_{i = j \leq R})_{1 \leq i \leq R, 1 \leq j \leq R}, \quad D^{\tau R} = (1_{i = j \leq R})_{1 \leq i \leq R, 1 \leq j \leq L}.$$ 

Note that $B^{\tau R, L - 1} = B$ and $D^{\tau R, L - 1} = D$.

**Proposition 2.11.** The price at time $t$ of a zero coupon of maturity $T$ is

$$P(t, T) = R e^{-r(T-t)} + (1-R) e^{-r(T-t)\gamma(X_{Z_{N^\tau_t}}, Z_{N^\tau_t}, t, T)H_t}$$

where

$$\gamma(X_{Z_{N^\tau_t}}, Z_{N^\tau_t}, t, T) = \frac{X_{Z_{N^\tau_t}}^T D^{\tau R}_{RZ_{N^\tau_t}} \exp \left( B^{\tau R_{RZ_{N^\tau_t}}} (t - Z_{N^\tau_t}) \right) D^{\tau R}_{RZ_{N^\tau_t}} \exp \left( B^{\tau R_{RZ_{N^\tau_t}}} (t - Z_{N^\tau_t}) \right) 1_{RZ_{N^\tau_t}}}{X_{Z_{N^\tau_t}}^T D^{\tau R}_{RZ_{N^\tau_t}} \exp \left( B^{\tau R_{RZ_{N^\tau_t}}} (t - Z_{N^\tau_t}) \right) 1_{RZ_{N^\tau_t}}}.$$ 

We now derive the dynamics of the zero coupon bond under the reduced filtration.

**Proposition 2.12.** Suppose that $t \in (t^{(p)}_i, t^{(p)}_{i+1})$ for some $i = 0, ..., n - 1$. The dynamics of the
bond price under $Q$ and with partial information is given by

$$dP(t, T) = rP(t, T)dt - (1-R) e^{-r(T-t)\gamma(X_{Z_{N^\tau_t}}, Z_{N^\tau_t}, t, T)H_t}$$

$$+ (1-R) e^{-r(T-t)} \left( \gamma(X_t, t, t, T) - \gamma(X_{Z_{N^\tau_t-1}}, Z_{N^\tau_t-1}, t, T) \right) \left( dN^\tau_t - \sum_{j > RZ_{N^\tau_t}} a_{R, j} H_t dt \right)$$

where $(\tilde{M}^0_0)$ and $\left( N^\tau_t - \int_0^t \sum_{j > RZ_{N^\tau_t}} a_{R, j} H_u du \right)$ are $(\mathcal{G}, Q)$-martingales.

The pricing of options in this model is more complex than in the previous models because the
density of the time of the last rating disclosure before the exercise date of the option is intricate.
However, it is possible to price an option when the exercise date of the option, $S$, corresponds to
a deterministic date of information disclosure $t^{(p)}_i$ for some $i = 1, ..., n$. 
Proposition 2.13. Let us denote by $z = Z_{N_t}$, the date of the last rating disclosure before time $t$. Assume that there exists $i = 1, \ldots, n$ such that $S = t^{(p)}_i$. The price of a call option of exercise date $S \geq t$ and of strike price $K$ is given by the following expression

$$C(t, S, K, T) = \sum_{j=1}^{L-1} \tilde{\delta}(e_j, X_z, z, t, S) e^{-r(S-t)} \left( e^{-r(T-S)} (R + (1 - R) \tilde{\gamma}(e_j, S, S, T)) - K \right)_+ H_t + (1 - \tilde{\gamma}(X_z, z, t, S) H_t) e^{-r(S-t)} \left( e^{-r(T-S)} R - K \right)_+$$

where $\tilde{\gamma}(.)$ is defined by Equation (2.7) and

$$\tilde{\delta}(e_j, X_z, z, t, S) = \mathbb{P}^Q (X_s = e_j | \mathcal{G}_t) = \sum_{i \leq R_z} e_j^\top \exp (A (S - t)) e_j X_z^\top D|_{R_z} \exp (B|_{R_z} (t - z)) D|_{R_z} e_j \mathbb{1}_{R_z}.$$

3 A model with contagion for two issuers

Pricing of names credit derivatives, such as recently studied by Herbertsson (2011) or Giesecke et al. (2011) requires to take into account dependence of default events. In this section, we extend the single name framework developed previously to introduce a contagion effect between two firms. More precisely, we consider two issuers $Y$ and $Z$ such that the probabilities of bankruptcy of each one is affected by the default of the other actor.

Both issuers are marked in a rating system counting $L$ levels. Information about current ratings is still carried by a continuous Markov chain $(X_t)$, defined on $(\Omega, \mathcal{F}, P)$. There exists exactly $L^2$ combinations of ratings. For this reason, $(X_t)$ takes now its values in a wider state space $E = \{e_1, e_2, \ldots, e_{L^2}\}$ that is the set of unit basis vector of $\mathbb{R}^{L^2}$. We will denote by $R^Y_t$ and $R^Z_t$ the ratings of $Y$ and $Z$ at time $t$ and we adopt the following convention

- if $X_t = e_j$ for $j = 1, \ldots, (L-1)^2$, then the ratings are equal to
  $$R^Y_t = \left\lfloor \frac{j}{(L-1)} \right\rfloor, \quad R^Z_t = j - \left\lfloor \frac{j}{(L-1)} \right\rfloor (L-1);$$

- if $X_t = e_j$ for $j = (L-1)^2 + 1, \ldots, (L-1)^2 + L - 1$, then the rating of $Y$ is equal to $L$ (Y is in default) and the rating of $Z$ is given by
  $$R^Z_t = j - (L-1)^2;$$

- if $X_t = e_j$ for $j = (L-1)^2 + L, \ldots, (L-1)^2 + 2L - 2$, then the rating of $Z$ is equal to $L$ (Z is in default) and the rating of $Y$ is given by
  $$R^Y_t = j - L(L-1);$$

- if $X_t = e_j$ for $j = L^2$, then both ratings are equal to $L$.

The sets of states in which $Y$ and $Z$ are in default are respectively given by

$$\Theta^Y = \{e_j \in E | j = (L-1)^2 + 1, \ldots, (L-1)^2 + L - 1 \text{ and } j = L^2\},$$
$$\Theta^Z = \{e_j \in E | j = (L-1)^2 + L, \ldots, L^2\}.$$

We denote by $\tau^Y = \inf \{t \geq 0 | X_t \in \Theta^Y\}$ and $\tau^Z = \inf \{t \geq 0 | X_t \in \Theta^Z\}$ the times of firms default. Of course we assume that at $t = 0$, both issuers are not in default such that $\mathbb{P}^Q (\tau^Y > 0, \tau^Z > 0) = 1$. 


1. The indicator variables, \( H^Y_t = 1_{\{t < r^Y\}} \) and \( H^Z_t = 1_{\{t < r^Z\}} \), are equal to one if issuers are still solvent.

We also define \( \Theta = (\Theta^Y)^c \cap (\Theta^Z)^c, \Theta^{YD} = \Theta^Y \cap (\Theta^Z)^c, \Theta^{ZD} = (\Theta^Y)^c \cap \Theta^Z \) and \( \Theta^{YZ} = \Theta^Y \cap \Theta^Z \). \( \Theta \) is the set of states where both issuers are still solvent, \( \Theta^{YD} \) (resp. \( \Theta^{ZD} \)) is the set of states where \( Y \) (resp. \( Z \)) has defaulted but \( Z \) (resp. \( Y \)) is still solvent, \( \Theta^{YZ} = \{e_{L,2}\} \) is the state where both firms have defaulted.

The pricing of bonds is done under a risk neutral measure \( Q \), under which the chain \( (X_t) \) is defined by a \( L^2 \times L^2 \) matrix of intensities of transition denoted by \( A = (a_{i,j})_{i,j=1,2,\ldots,L} \). The matrix of intensities of transitions takes the following form

\[
A = \begin{bmatrix}
B^{(1,1)} & B^{(1,2)} & B^{(1,3)} & b^{(1)} \\
0_{(L-1),(L-1)} & B^{(2,2)} & 0_{(L-1),(L-1)} & b^{(2)} \\
0_{(L-1),(L-1)} & 0_{(L-1),(L-1)} & B^{(3,3)} & b^{(3)} \end{bmatrix}
\]

where \( B^{(i,j)} \) are submatrices of \( A \) such that \( B^{(1,1)} = A|_{\Theta \rightarrow \Theta}, B^{(1,2)} = A|_{\Theta \rightarrow \Theta^{YD}}, B^{(1,3)} = A|_{\Theta \rightarrow \Theta^{ZD}}, B^{(2,2)} = A|_{\Theta^{YD} \rightarrow \Theta^{YD}}, B^{(2,3)} = A|_{\Theta^{ZD} \rightarrow \Theta^{YD}} \) and \( b^{(j)} \) are the following vectors

\[
b^{(1)} = A|_{\Theta \rightarrow \Theta^{YD}} = - (B^{(1,1)} + B^{(1,2)} + B^{(1,3)}) 1_{(L-1)^2};
b^{(2)} = A|_{\Theta^{YD} \rightarrow \Theta^{YD}} = -B^{(2,2)} 1_{(L-1)}; 
b^{(3)} = A|_{\Theta^{ZD} \rightarrow \Theta^{YD}} = -B^{(3,3)} 1_{(L-1)}.
\]

We provide in section 4.4 a detailed example in which the scale of ratings counts three levels so as to visualize the structure of the matrix \( A \) in a simple setting. We also denote \( B \) and \( b \) the following submatrices of \( A \)

\[
B = \begin{bmatrix}
B^{(1,1)} & B^{(1,2)} & B^{(1,3)} \\
0_{(L-1),(L-1)} & B^{(2,2)} & 0_{(L-1),(L-1)} \\
0_{(L-1),(L-1)} & 0_{(L-1),(L-1)} & B^{(3,3)} \end{bmatrix}, \quad b = \begin{bmatrix}
b^{(1)} \\
b^{(2)} \\
b^{(3)} \end{bmatrix}
\]

and let \( G^{(1)} \) (resp. \( G^{(2)} \)) be a diagonal \( (L^2 - 1) \times (L^2 - 1) \) matrix whose \( i \)-th diagonal element for \( i = 1, \ldots, L^2 - 1 \) equals 1 if \( e_i \in (\Theta^Y)^c \) (resp. \( (\Theta^Z)^c \)) and is 0 otherwise. Finally to characterize the dynamics of ratings of \( Y \) or \( Z \) when the other issuer has defaulted, we introduce the following notation

\[
D^{(2)} = \begin{bmatrix}
0_{(L-1),(L-1)} & 1d_{(L-1)} & 0_{(L-1),L} \\
0_{(L-1),(L-1)} & 1d_{(L-1)} & 0_{(L-1),1} \\
0_{1,(L-1)} & b^{(2)} & 0_{1,(L-1)} \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix}
B^{(2,2)} & b^{(2)} \\
0 & 0_{1,(L-1)} & 0 \end{bmatrix}
\]

and

\[
A^{(3)} = \begin{bmatrix}
B^{(3,3)} & b^{(3)} \\
0 & 0_{1,(L-1)} & 0 \end{bmatrix}
\]

We now assume that the credit quality of issuers is only assessed at discrete (and non random) times \( 0 = t_0 \leq t_1 \leq \ldots \leq t_n = T \). The information available to the market is represented by the filtration \( \{\mathcal{G}_t, t \leq T\} \) where the sigma algebra \( \mathcal{G}_t \) at time \( t \in [t_i,t_{i+1}) \) is now defined by \( \mathcal{G}_t = \sigma \{X_{t_0}, \ldots, X_{t_i}, H^Y_{u}, H^Z_{u}, u \leq t\} \).

**Proposition 3.1.** Suppose that \( t \in [t_i,t_{i+1}) \) for some \( i = 0, \ldots, n-1 \), then the price of bonds of maturity \( T \) issued by \( Y \) and \( Z \) are given by the following expressions.

\[
P^Y(t,T) = Re^{-r(T-t)} + (1-R)e^{-r(T-t)} P^Q(\tau^Y > T \mid \mathcal{G}_t),
\]

\[
P^Z(t,T) = Re^{-r(T-t)} + (1-R)e^{-r(T-t)} P^Q(\tau^Z > T \mid \mathcal{G}_t),
\]
where
\[ P^Q(\tau^Y > T \mid \mathcal{G}_t) = \gamma^Y(X_{t_i}, t, t, T) H_t^Y H_t^Z + \gamma^Y_S(X_{t_i}, t, t, \tau_Z, T) H_t^Y (1 - H_t^Z) \]
\[ P^Q(\tau^Z > T \mid \mathcal{G}_t) = \gamma^Z(X_{t_i}, t, t, T) H_t^Z H_t^Y + \gamma^Z_S(X_{t_i}, t, t, \tau_Y, T) H_t^Z (1 - H_t^Y) \]

and
\[ \gamma^Y(X_{t_i}, t, t, T) = P^Q(\tau^Y > T \mid X_{t_i}, t < \tau^Y, t < \tau^Y) = \frac{X_t^\top D_t^\top \exp(B(t - t_i)) G^{(2)} \exp(B(T - t)) G^{(1)} 1_{L^2-1}}{X_{t_i}^\top D_{t_i}^\top \exp(B(t - t_i)) G^{(2)} G^{(1)} 1_{L^2-1}}, \tag{3.1} \]
\[ \gamma^Z(X_{t_i}, t, t, T) = P^Q(\tau^Z > T \mid X_{t_i}, t < \tau^Z, t < \tau^Y) = \frac{X_t^\top D_t^\top \exp(B(t - t_i)) G^{(1)} \exp(B(T - t_i)) G^{(2)} 1_{L^2-1}}{X_{t_i}^\top D_{t_i}^\top \exp(B(t - t_i)) G^{(1)} G^{(2)} 1_{L^2-1}}, \tag{3.2} \]
\[ \gamma^Y_S(X_{t_i}, t, t, \tau_Z, T) = P^Q(\tau^Y > T \mid X_{t_i}, t \geq \tau^Z = t_z, t < \tau^Y) = \frac{\sum_{e_j \in \Theta} X_{t_i}^\top \exp(A(t_z - t_i)) e_j e_j^\top (D(3))^\top \exp(B(3) (T - t_i)) 1_{L^2-1}}{\sum_{e_j \in \Theta} X_{t_i}^\top \exp(A(t_z - t_i)) e_j e_j^\top (D(3))^\top \exp(B(3) (t_z - t)) 1_{L^2-1}}, \tag{3.3} \]
\[ \gamma^Z_S(X_{t_i}, t, t, \tau_Y, T) = P^Q(\tau^Z > T \mid X_{t_i}, t < \tau^Z, t > \tau^Y = t_y) = \frac{\sum_{e_j \in \Theta} X_{t_i}^\top \exp(A(t_y - t_i)) e_j e_j^\top (D(2))^\top \exp(B(2) (T - t_y)) 1_{L^2-1}}{\sum_{e_j \in \Theta} X_{t_i}^\top \exp(A(t_y - t_i)) e_j e_j^\top (D(2))^\top \exp(B(2) (t_y - t)) 1_{L^2-1}}. \tag{3.4} \]

In this setting, we are also able to calculate the correlation between default events.

**Proposition 3.2.** Suppose that \( t \in [t_i, t_{i+1}] \) for some \( i = 0, ..., n - 1 \), then the correlation between default events are provided by
\[
\text{Cor} \left( 1_{\{\tau^Y < T\}, 1_{\{\tau^Z < T\}} \mid X_{t_i}, \tau^Z > t, \tau^Y > t \right) = \frac{\gamma^Y Z - \gamma^Y \gamma^Z}{\sqrt{\gamma^Y \gamma^Z (1 - \gamma^Y) (1 - \gamma^Z)}},
\]
where \( \gamma^Y = \gamma^Y(X_{t_i}, t, t, T) \) and \( \gamma^Z = \gamma^Z(X_{t_i}, t, t, T) \) are defined by Equation (3.1) and (3.2), and
\[
\gamma^Y Z = \gamma^Y(Z(X_{t_i}, t, t, T)) = \frac{X_t^\top D_t^\top \exp(B(T - t_i)) G^{(1)} G^{(2)} 1_{L^2-1}}{X_{t_i}^\top D_{t_i}^\top \exp(B(t - t_i)) G^{(1)} G^{(2)} 1_{L^2-1}}.
\]

**Remark 3.3.** The pricing of options in the contagion model is much more complex than in model with a single entity. However, it is possible to price an option on a basket of two zero coupon bonds with weights \((\omega_Y, \omega_Z)\) when both actors issue zero coupon bonds of same maturity and when the exercise date of the option, \( S \), corresponds to a date of information disclosure \( t_S \). Suppose that \( t \in [t_i, t_{i+1}] \) for some \( i = 0, ..., n - 1 \). The price of a call option of exercise date \( S \) and of strike price \( K \) is given by the next expression (see Section (6.13))
\[
C(t, S, K, T) = \sum_{e_i \in \Theta} P^Q(X_{t_S} = e_i \mid \mathcal{G}_t) e^{-(S - t)} \left[ (\omega_Y + \omega_Z) R + (1 - R) \omega_Y \gamma^Y + (1 - R) \omega_Z \gamma^Z - K \right] +
+ \sum_{e_i \in \Theta^{ZD}} P^Q(X_{t_S} = e_i \mid \mathcal{G}_t) e^{-(S - t)} \int_{t}^{t_S} \left[ (\omega_Y + \omega_Z) R + (1 - R) \omega_Y \gamma^Y_S (t_z) - K \right] + f_{ZD}(t_z) dt_z
+ \sum_{e_i \in \Theta^{YD}} P^Q(X_{t_S} = e_i \mid \mathcal{G}_t) e^{-(S - t)} \int_{t}^{t_S} \left[ (\omega_Y + \omega_Z) R + (1 - R) \omega_Z \gamma^Z_S (t_y) - K \right] + f_{YD}(t_y) dt_y
+ \sum_{e_i \in \Theta^{YZ}} P^Q(X_{t_S} = e_i \mid \mathcal{G}_t) e^{-(S - t)} \left[ (\omega_Y + \omega_Z) R - K \right] \tag{3.5}
\]
4 Numerical applications

4.1 Model with predetermined times of information disclosure

In the first part of this section, we show how the deficit of information can be taken into account for the calculation of the risk neutral transition probabilities of a AA sovereign issuer such the French republic. We have retrieved the historical matrix of sovereign transition probabilities computed by Standard & Poors, on the period 1993-2010 and obtained the intensities under $P$ as the matrix logarithm of this matrix. We assume in the sequel that France has the same transition matrix under the risk neutral measure and use it the following tests.

Note that at the time of writing this paper, the last publication of information dates from the 13th of January 2012. S&P downgraded France from AAA to AA+ and correspond to a delay of information equal to 0.2219. In the remainder of this section, we will try to understand how the deficit of information affects prices of bonds and options. For this purpose, we compare zero coupon bonds prices for each rating and for maturities that range from 1 to 20 years. In our first set of tests, we assume that the rating has been disclosed for the last time, a half year ago $(t_i = -0.5)$ and the interest rate is set to $r = 2\%$. A credit spread is retrieved by inverting the formula:

$$P(t_i = -0.5)(0, T) = e^{(r + spread(t_i = -0.5))T}. \tag{4.1}$$

In a second set of tests, bond prices are computed under the assumption that information about ratings is available at the date of calculation (which is equivalent to set $t_i = 0$). Credit spreads are in this case noted $spread(t_i = 0)$. The part of the credit spread related to the deficit of information is then calculated as the following difference:

$$spread\text{ attributed to information} = spread(t_i = -0.5) - spread(t_i = 0)$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
 & AAA & AA & A & BBB & BB & B & CCC/CC & D \\
\hline
AAA & 0.9742 & 0.0258 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
AA & 0.0212 & 0.9048 & 0.0553 & 0.0551 & 0.0459 & 0.1541 & 0.0561 & 0.0000 \\
A & 0.0000 & 0.2560 & 0.9194 & 0.0549 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
BBB & 0.0000 & 0.0000 & 0.0000 & 0.0459 & 0.8673 & 0.0561 & 0.0139 & 0.0154 \\
BB & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
B & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
CCC/CC & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.4737 & 0.1052 \\
D & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1052 & 1.0000 \\
\hline
\end{tabular}

Table 4.1: Probabilities of transition, 1 year horizon.

where $\gamma^Y = \gamma^Y(e_i, t\bar{s}, t\bar{s}, T)$, $\gamma^Z = \gamma^Z(e_i, t\bar{s}, t\bar{s}, T)$, $\gamma^{YS}(t_z) = \gamma^{YS}(e_i, t\bar{s}, t\bar{s}, t_z, T)$, $\gamma^{ZS}(t_y) = \gamma^{ZS}(e_i, t\bar{s}, t\bar{s}, t_y, T)$,

$$p^Q(X_{t\bar{s}} = e_1 | \mathcal{G}_t) = \frac{X_t^\top \exp(A(t_{t\bar{s}} - t_i)) e_j}{\sum_{e_k \in \Theta} X_t^\top \exp(A(t - t_i)) e_k}$$

and

$$f_{ZD}(u) = f_{\tau_u < \tau_s \leq t_s(u)}$$

$$f_{ZD}(u) = \frac{X_t^\top D^\top \exp(B(t - t_i)) G(1) \exp(B(u - t_i)) B G(2)}{X_t^\top D^\top \exp(B(t - t_i)) G(1) \exp(B(t - t_i)) B G(2) - X_t^\top D^\top \exp(B(t - t_i)) G(1) G(2)}$$

$$f_{YD}(t_y) = f_{\tau_y < \tau_s \leq t_s(t_y)}$$

$$f_{YD}(t_y) = \frac{X_t^\top D^\top \exp(B(t - t_i)) G(2) \exp(B(u - t)) B G(1)}{X_t^\top D^\top \exp(B(t - t_i)) G(2) \exp(B(t - t_i)) G(1) - X_t^\top D^\top \exp(B(t - t_i)) G(2) G(1)}$$

where $\gamma^Y = \gamma^Y(e_i, t\bar{s}, t\bar{s}, T)$, $\gamma^Z = \gamma^Z(e_i, t\bar{s}, t\bar{s}, T)$, $\gamma^{YS}(t_z) = \gamma^{YS}(e_i, t\bar{s}, t\bar{s}, t_z, T)$, $\gamma^{ZS}(t_y) = \gamma^{ZS}(e_i, t\bar{s}, t\bar{s}, t_y, T)$,
The evolution of the part of spreads attributable to the information delay is displayed in the left
graph of Figure 4.1, for bonds issued by AA, A and BBB issuers. We observe that the deficit of
information does not have the same impact on every rating. For AA and A bonds, the part of
spreads attributable to the information is positive and increasing with the maturity. This trend
can be explained by the fact that a half year after the last rating disclosure, the country can
be downgraded with non negligible probabilities. More surprising, the fraction of credit spreads
linked to the lack of information is decreasing for BBB issuers. This is probably related to the
fact that if the country is not in default after 6 months, the probability of being in a higher rating
is high and credit spreads are then lower than if the rating was disclosed today.

Finally, we have tested the influence of the delay between the last disclosure of information and
the date of bonds issuance. The left graph of figure 4.1 presents the part of spreads attributable
to the information delay, for different time lag: \( t_i = -0.5, -1, -1.5 \) and for A rated bonds. For
this category of bonds, the higher is the delay of disclosure, the higher is the cost of this lack of
information.

We have priced call options on a zero coupon bond of maturity 10 years and of rating B. The
recovery rate is still null. The exercise date is set to \( S = 5 \) years and strikes ranges from 0.65 to
0.90. The left graph of figure 4.2 presents options prices for \( t_i = -4 \) to -1 years and \( t_S = 4 \) years.
The right graph of figure 4.2 presents options prices for \( t_i = 0 \) years and \( t_S = 2 \) to 3 years. We
clearly see in both cases that the higher is the deficit of information, the higher are options prices.

4.2 Model with information at exogenous random dates.

In this section, we test the model in which the information about the real rating of the issuer is
published at random times. As in the previous paragraph, we work with the matrix of sovereign
transition rates of table 4.1. The formula of bond prices being identical to the one of the previous
model, we limit our analyze to option prices. Figure 4.3 presents prices of call options on a zero
coupon bond of maturity 10 years and of ratings B and A. The recovery rate is still null. The
exercise date is set to \( S = 5 \) years and strikes ranges from 0.65 to 0.90. The left graph of figure 4.3
presents options prices for a frequency of information disclosure equal to \( \lambda = 1, 2, 3 \) years. For both
ratings, a low frequency of information release rises the prices, whatever the strike prices. In figure
4.4, we display three simulated trajectories of a bond price (maturity 10 years and rating AA)
under the risk neutral measure. These trajectories are computed by discretization of equation
Influence of the delay of information on option prices

\[ t_i = -4 \]

\[ t_i = -3 \]

\[ t_i = -2 \]

\[ t_i = -1 \]

\[ t_S = 2 \]

\[ t_S = 3 \]

\[ t_S = 4 \]

\[ t_S = 5 \]

Figure 4.2: Options prices.

(2.6). The jumps are related to the arrival of a new information and the amplitude of jumps depends on the new rating disclosed.

4.3 Model with information at endogenous random dates.

In this section, we assess numerically the credit spread of bonds, when the information is disclosed at fixed times and when a credit worsening occurs. As mentioned earlier, this assumption introduces asymmetry in the information arrival process. We still work with the matrix of sovereign transition rates of table 4.1. Figure 4.5 presents the credit spreads, such as defined by (4.1), for a BBB zero coupon bond. The risk free rate is 2% and two scenarios are considered: when the last rating has been published one and two years ago. We also reported on the figure, the spreads of BBB bond, when the information is not endogenous. We observe that spreads are lower when the information is endogenous. This observation can be justified as follows: when the information is fully exogenous, the probability that the country has in fact a lower rating than the last one reported, is not null. It is not the case when the information is endogenous: in case of worsening of the economic situation of the country, the rating is adjusted and disclosed immediately. The fact that the country has not been downgraded since one or two years, gives us an indication that the country has the same or a higher rating. Similar conclusions can be drawn for options prices: the endogeneity attenuates the impact of the delay of information.

4.4 Model with contagion.

So as to present explicitly the matrix \( A \), we work under the assumption that the scale of ratings count three levels and that the issuers \( Y \) and \( Z \) cannot default at the same time. Furthermore, we assume that, when both issuers are in activity, matrix of intensities, ruling the marginal evolution of ratings, are identical and such that the one year probabilities of transition are:

\[
e^A = \exp \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.05 & 0.90 & 0.05 \\ 0.10 & 0.80 & 0.10 \\ 0 & 0 & 1 \end{pmatrix}
\]

If \( Z \) is in default, then the transition intensities of \( Y \) are multiplied by a positive constant \( \alpha \). If \( \alpha \) strictly is bigger than one, the probabilities that \( Y \) changes of rating or goes to bankruptcy are
Figure 4.3: Options prices, influence of disclosure frequencies.

Figure 4.4: Evolution of bond prices.

Figure 4.5: Comparison of credit spreads with and without endogeneity
then increased. The parameter $\alpha$ allows us to model the contagion effect of the default of $Z$ on the rating of $Y$. In case of bankruptcy of $Y$, the transition intensities of $Z$ are multiplied by a positive constant $\beta$. If $\beta$ differs from $\alpha$, the contagion effect is asymmetric. The transition matrix $A$ of the system is in this case equal to:

$$
X_t \rightarrow (R_Y, R_Z) \rightarrow (1, 1) \rightarrow (1, 2) \rightarrow (2, 1) \rightarrow (2, 2) \rightarrow (3, 1) \rightarrow (3, 2) \rightarrow (1, 3) \rightarrow (2, 3) \rightarrow (3, 3)
$$

In the following tests, we have set $\alpha = 0.8$ and $\beta = 1.2$. The risk free rate is still equal to 2% while the recovery rate is null. When required, the default times $t_z$ and $t_y$ have been set to $-0.5$. The last disclosure is at the date of the issuance, $t_i = 0$. Figure 4.6 exhibits the difference between credit spreads computed with and without contagion (situation in which $\alpha = 1$ and $\beta = 1$). As we could expect, for the chosen values of $\alpha$ and $\beta$, if $Y$ collapses before $Z$, the probability of bankruptcy of this last issuer decreases and the spreads follow the same trend. In the opposite case, the default of $Z$ directly raises the probability of default of $Y$ and increases the spreads.

In figure 4.7, we plot the correlation between the one year default times ($1_{T^Y < T}$ and $1_{T^Z < T}$), for different levels of contagion. $\alpha$ is set to 1, while $\beta$ varies from 0.5 to 3. Two couples of rating are considered: $(R_Y, R_Z) =$ (1,1) and (2,2). The graph reveals that the rating directly affects the level of correlation of default times. When $\beta = 1$, default times are independent. For lower or higher $\beta$’s, the correlation is respectively negative or positive. We also observe that the deficit of information directly influences the correlation. If the last rating disclosure has been made 3 years ago, the correlation is higher (for $(R_Y, R_Z) =$ (1,1)) or lower (for $(R_Y, R_Z) =$ (2,2)) than in case of a recent disclosure. Figure 4.8 illustrates the influence of the information delay on the price of a basket option for different strikes. The exercise date is set to 5 years and the maturity of both bonds is 10 years. The risk free and recovery rates are respectively 2% and 0%. The ratings are set to $(R_Y, R_Z) =$ (2,2) and the weights of the basket are $\omega_Y = \omega_Z = 50\%$. As in previous section, a lack of recent information raises the option prices.
Figure 4.7: Influence of the delay of disclosure on correlation.

Figure 4.8: Influence of the delay of disclosure on basket option prices.
5 Conclusions.

This paper explores a new use of Phase-type distributions in credit risk under different flows of incomplete information. In particular, it details the influence of discontinuity and delay of information on prices of credit linked securities, in a credit rating system. In the first model considered, information about the rating arrives at predetermined dates. In this framework, we infer prices of zero coupon bonds, their dynamics in continuous time, and prices of call options. Numerical applications reveal that a deficit of information raises most of the time credit spreads and option prices but the impact varies considerably across ratings. In the second model studied, information is disclosed at random times. This assumption is more realistic and affects mainly the dynamics of bonds and option prices. In the third model ratings is disclosed at non random dates and the credit worthiness of the firm or of the country becomes worse than the last known rating could let to predict. This asymmetry in the information arrival process reduces the impact of information delay on credit spreads. In the last model presented, we propose a contagion model for two issuers, when information flows at fixed dates. Numerical applications show us that a deficit of information affects either positively or negatively the correlation between default times.

6 Proofs

6.1 Proof of Proposition 2.1

Equation (2.2) can be developed as follows

\[ P(t,T) = e^{-r(T-t)} \mathbb{E}^Q \left( R + (1-R) H_T \big| X_{t_0}, \ldots, X_{t_i}, H_u, u \leq t \right) \]

\[ = R e^{-r(T-t)} \mathbb{E}^Q \left( H_T \big| X_{t_0}, \ldots, X_{t_i}, H_t = 1 \right) H_t, \]

and according to the Bayes’ rule

\[ \mathbb{E}^Q \left( H_T \big| X_{t_0}, \ldots, X_{t_i}, H_t = 1 \right) = \frac{P^Q(\tau > T \big| X_{t_i}, t < \tau)}{P^Q(\tau > t \big| X_{t_i}, t_i < \tau)} \]

\[ = \frac{X_{t_i}^{\top} D^{\top} \exp \left( B (T - t_i) \right) 1_{L-1}}{X_{t_i}^{\top} D^{\top} \exp \left( B (t - t_i) \right) 1_{L-1}}. \]

6.2 Proof for Equation 2.4

The price of the defaultable bond is given by

\[ P(t,T) = \mathbb{E}^Q \left( e^{-r(T-t)} 1_{\{\tau > T\}} + e^{-r(\tau-t)} R 1_{\{\tau \leq T\}} \big| \mathcal{G}_t \right) \]

\[ = R \mathbb{E}^Q \left( e^{-r(\tau-t)} \big| \mathcal{G}_t \right) + e^{-r(T-t)} \mathbb{E}^Q \left( \left( 1 - R e^{-r(\tau-T)} \right) H_T \big| \mathcal{G}_t \right) \]

\[ = R \mathbb{E}^Q \left( e^{-r(\tau-t)} \big| \mathcal{G}_t \right) + e^{-r(T-t)} \mathbb{E}^Q \left( H_T \big| \mathcal{G}_t \right) - R e^{-r(T-t)} \mathbb{E}^Q \left( e^{-r(\tau-T)} H_T \big| \mathcal{G}_t \right). \]

The first expectation can be developed as follows

\[ \mathbb{E}^Q \left( e^{-r(\tau-t)} \big| \mathcal{G}_t \right) = \mathbb{E}^Q \left( e^{-r(\tau-t)} \big| X_{t_0}, \ldots, X_{t_i}, H_u, u \leq t \right) \]

\[ = e^{r(t-t)} (1 - H_t) + \mathbb{E}^Q \left( e^{-r(\tau-t)} \big| X_{t_i}, H_t = 1 \right) H_t \]

\[ = e^{r(t-t)} (1 - H_t) + \frac{\mathbb{E}^Q \left( e^{-r(\tau-t)} 1_{\{\tau > t_i\}} \big| X_{t_i}, \tau > t_i \right)}{P^Q(\tau > t \big| X_{t_i}, \tau > t_i)} H_t \]

The Laplace transform of \( \tau \) is given by (see e.g. Rolski et al. (2009))

\[ \mathbb{E}^Q \left( e^{-\omega(\tau-t)} \big| X_t \right) = X_t^{\top} D^{\top} \left( \omega I_{L-1} - B \right)^{-1} b \]
Finally, the expectation can be rewritten as follows

\[
\mathbb{E}^Q \left( e^{-r(\tau-t)}1_{\{\tau > t\}} | X_t, \tau > t \right) = \sum_{j=1}^{L-1} \mathbb{E}^Q \left( e^{-r(\tau-t)}1_{\{X_t = e_j\}} | X_t, \tau > t \right) = \sum_{j=1}^{L-1} \mathbb{E}^Q \left( e^{-r(\tau-t)}|X_t = e_j\right) P^Q (X_t = e_j | X_t, \tau > t) = \sum_{j=1}^{L-1} e_j D^\top (rI_{L-1} - B)^{-1}bX_t^\top \exp(A(t-t_i))e_j.
\]

and therefore

\[
\mathbb{E}^Q \left( e^{-r(\tau-t)} | \mathcal{G}_t \right) = e^{r(t-\tau)} (1 - H_t) + \sum_{j=1}^{L-1} \frac{e_j D^\top (rI_{L-1} - B)^{-1}bX_t^\top \exp(A(t-t_i))e_j}{X_t^\top D^\top \exp(B(t-t_i))1_{1(L-1)}} H_t.
\]

The second expectation has already been calculated in the proof of Proposition 2.1, and is equal to

\[
\mathbb{E}^Q (H_T \mid \mathcal{G}_t) = \frac{X_t^\top D^\top \exp(B(T-t_i))1_{1(L-1)}}{X_t^\top D^\top \exp(B(t-t_i))1_{1(L-1)}} H_t.
\]

The last expectation is calculated as follows:

\[
\mathbb{E}^Q \left( e^{-r(T-t)}H_T \mid \mathcal{G}_t \right) = \mathbb{E}^Q \left( e^{-r(T-t)}1_{\{\tau > T\}} | X_t, H_t = 1 \right) H_t = H_t \sum_{j=1}^{L-1} \mathbb{E}^Q \left( e^{-r(T-t)}1_{\{X_T = e_j\}} | X_t, H_t = 1 \right) = H_t \sum_{j=1}^{L-1} \mathbb{E}^Q \left( e^{-r(T-t)}|X_T = e_j\right) P^Q (X_T = e_j | X_t, \tau > T) = H_t \sum_{j=1}^{L-1} \mathbb{E}^Q \left( e^{-r(T-t)}|X_T = e_j\right) P^Q (\tau > t | X_t, t_i < \tau) = H_t \sum_{j=1}^{L-1} e_j D^\top (rI_{L-1} - B)^{-1}b \frac{X_t^\top \exp(A(T-t_i))e_j}{X_t^\top D^\top \exp(B(t-t_i))1_{1(L-1)}}.
\]

6.3 Proof of Proposition 2.3

By Proposition 2.1, we get that for \( t \in (t_i^{(p)}, t_{i+1}^{(p)}) \)

\[
dP(t, T) = rP(t, T)dt + (1 - R)e^{-r(T-t)} \frac{\partial}{\partial t} \gamma(X_t, t_i, t, T) H_t dt + (1 - R)e^{-r(T-t)} \gamma(X_t, t_i, t, T) dH_t.
\]

First, note that, according to the definition of \( \gamma \), its derivative is equal to

\[
\frac{\partial}{\partial t} \gamma(X_t, t_i, t, T) = - \frac{X_t^\top D^\top \exp(B(T-t_i))1}{X_t^\top D^\top \exp(B(t-t_i))1} X_t^\top D^\top \exp(B(t-t_i))B 1_{1(L-1)} = - \gamma(X_t, t_i, t, T) \frac{X_t^\top D^\top \exp(B(t-t_i))B 1_{1(L-1)}}{X_t^\top D^\top \exp(B(t-t_i))1_{1(L-1)}}.
\]

Second, let us characterize the martingale representation of \( H_t \) under the market filtration \( \mathcal{G} \). Recall that the rating \( R_t \) is equal to the scalar product of \( X_t \) and of the \( L \) dimensional vector
We know that $H_t^0 = 1_{\{\tau \leq t\}}$ is a stopped jump process that can be written as

$$H_t^0 = \int_0^t a_{R_u,L} H_u du + M_t^0$$

where $M_t^0$ is a real martingale on the enlarged filtration $(\mathcal{F}, Q)$ (see e.g. Lemma 11.2.3 in Bielecki and Rutkowski (2002)).

Let us now recall that:

1. For every $(\mathcal{F}, Q)$ martingale $M_t$, its optional projection $\mathbb{E} (M_t | G_t)$ is a $(\mathcal{G}, Q)$ martingale.

2. For any progressively measurable process $\psi_t$ with $\mathbb{E}^Q \left( \int_0^T |\psi_u| du \right) < \infty$, the difference

$$\mathbb{E}^Q \left( \int_0^t \psi_u du | G_t \right) - \int_0^t \mathbb{E}^Q (\psi_u | G_u) du, \quad t \leq T,$$

is a $(\mathcal{G}, Q)$ martingale.

We can then infer that

$$\hat{M}_t^0 = \mathbb{E}^Q (H_t^0 | G_t) - \int_0^t \mathbb{E}^Q (a_{R_u,L} H_u | G_u) du = H_t^0 - \int_0^t \mathbb{E}^Q (a_{R_u,L} | G_u) H_u du. \quad (6.2)$$

is a $(\mathcal{G}, Q)$ martingale. Moreover we have

$$\mathbb{E}^Q (a_{R_t,L} | G_t) = \mathbb{E}^Q (a_{R_{t-},L} | X_{t_0}, \ldots, X_{t-}, H_t = 1) H_t$$

$$= \sum_{j=1}^{L-1} a_{j,L} H_t P^Q (X_t = e_j | X_{t_i}, t < \tau),$$

and the probability that the issuer has the rating $j$ at time $t$ is calculated by the Bayes’ rule

$$P^Q (X_t = e_j | X_{t_i}, t < \tau) = \frac{P^Q (X_t = e_j | X_{t_i}, t_i < \tau)}{P^Q (\tau > t | X_{t_i}, t_i < \tau)} = \frac{X_{t_i}^\top \exp (A (t - t_i)) e_j}{X_{t_i}^\top D^\top \exp (B (t - t_i)) 1_{L-1}}.$$

Combining equations (6.2) and (6.1) allows us to rewrite the dynamics of the zero coupon bond price as follows

$$dP(t,T) = rP(t,T)dt - (1-R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t,T) X_{t_i}^\top D^\top (\exp (B (t - t_i)) B) 1_{L-1} - H_t dt$$

$$- (1-R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t,T) \sum_{j=1}^{L-1} a_{j,L} X_{t_i}^\top \exp (A (t - t_i)) e_j H_t dt$$

$$- (1-R)e^{-r(T-t)}\gamma(X_{t_i}, t_i, t,T) d\hat{M}_t^0.$$

Third, it remains to prove that

$$X_{t_i}^\top D^\top (\exp (B (t - t_i)) B) 1_{L-1} = - \sum_{j=1}^{L-1} a_{j,L} X_{t_i}^\top \exp (A (t - t_i)) e_j \quad (6.3)$$

to conclude. According to the definition of $A$, we know that (see Rolski et al. (2009)) that

$$\exp(At) = \begin{pmatrix} \exp(tB) & 1_{L-1} - \exp(tB)1_{L-1} \\ 0 & 1 \end{pmatrix}.$$

Then, for $j \neq L$, we have that

$$X_{t_i}^\top \exp (A (t - t_i)) e_j = X_{t_i}^\top D^\top \exp (B (t - t_i)) D e_j.$$
By construction, the vector $b$ is the vector with components $a_{j,L}$ for $j = 1 \ldots L - 1$ and it satisfies $b = -B 1_{L-1}$. Therefore
\begin{align*}
- \sum_{j=1}^{L-1} a_{j,L} X_{t_i}^\top \exp (A (t - t_i)) e_j &= -X_{t_i}^\top D^\top \exp (B (t - t_i)) b \\
&= X_{t_i}^\top D^\top (\exp (B (t - t_i)) B) 1_{L-1},
\end{align*}
which ends the proof.

### 6.4 Proof of Proposition 2.5

According to Proposition 2.1, the price of the call can be rewritten as follows
\begin{align*}
C(t, S, K, T) &= \mathbb{E}^Q \left( e^{-r(T-t)} (P(S, T) - K)_+ | G_t \right) \\
&= e^{-r(T-t)} \mathbb{E}^Q \left( \left( e^{-r(T-S)} (R + (1 - R)\gamma(X_{t_S}, t_S, S, T)H_S) - K \right)_+ | G_t \right) \\
&= e^{-r(T-t)} \mathbb{E}^Q \left( \left( e^{-r(T-S)} [R + (1 - R)\gamma(X_{t_S}, t_S, S, T)H_S] - K \right)_+ | X_{t_0}, \ldots, X_{t_t}, H_t = 1 \right) H_t \\
&\quad + (1 - H_t) e^{-r(T-t)} \left( e^{-r(T-S)} R - K \right)_+.
\end{align*}

The expectation in the second term of equation (6.4) is equal to
\begin{align*}
\mathbb{E}(H_S | X_{t_0}, \ldots, X_{t_t}, H_t = 1) &= P^Q (\tau > S | X_{t_i}, t < \tau, t_i < \tau) = \gamma(X_{t_i}, t_i, t, S).
\end{align*}

So as to lighten future calculations, we use the notation
\begin{align*}
g(X_{t_S}) &= \left( e^{-r(T-S)} (R + (1 - R)\gamma(X_{t_S}, t_S, S, T)) - K \right)_+.
\end{align*}

The expectation in the first term of Equation (6.4) becomes then
\begin{align*}
&\mathbb{E}^Q \left( H_S \left[ \left( e^{-r(T-S)} (R + (1 - R)\gamma(X_{t_S}, t_S, S, T)) - K \right)_+ \right] | X_{t_0}, \ldots, X_{t_t}, H_t = 1 \right) \\
&= \mathbb{E}^Q \left( 1_{\{X_S \neq e_L\}} g(X_{t_S}) | X_{t_0}, \ldots, X_{t_t}, t < \tau \right) \\
&= \sum_{j=1}^{L-1} g(e_j)P^Q (\tau > S, X_{t_S} = e_j | X_{t_0}, \ldots, X_{t_t}, t < \tau) \\
&= \sum_{j=1}^{L-1} g(e_j)P^Q (\tau > S | X_{t_S} = e_j, t < \tau) P^Q (X_{t_S} = e_j | X_{t_i}, t < \tau).
\end{align*}

Given that the default state is absorbing, we have for $j \neq L$ that
\begin{align*}
P^Q (X_{t_S} = e_j | X_{t_i}, t < \tau) &= P^Q (X_{t_S} = e_j, \tau > t | X_{t_i}, t_i < \tau) \\
&= \frac{P^Q (X_{t_S} = e_j, \tau > t | X_{t_i}, t_i < \tau)}{P^Q (\tau > t | X_{t_i}, t_i < \tau)} \\
&= \frac{X_{t_i}^\top \exp (A (t_S - t_i)) e_j}{X_{t_i}^\top D^\top \exp (B (t - t_i)) 1_{L-1}}
\end{align*}

and
\begin{align*}
P^Q (\tau > S | X_{t_S} = e_j, t < \tau) &= e_j^\top D^\top \exp (B (S - t_S)) 1_{L-1}.
\end{align*}
6.5 Proof of Proposition 2.7

The proof is similar to the proof of Proposition 2.3, excepted that we have now a jump component related to the disclosure of information. If the last date of rating disclosure before \( t \) is \( Z_{N_{r}^{-}} = z \), then according to the bond price formula, the differential of \( P(t, T) \) is given by

\[
dP(t, T) = rP(t, T)dt + (1 - R)e^{-r(T-t)} \frac{\partial}{\partial t} \gamma(X_{z}, z, t, T) H_{t}dt + (1 - R)e^{-r(T-t)} \gamma(X_{z}, z, t, T) dH_{t} + (1 - R)e^{-r(T-t)} (\gamma(X_{t}, t, t, T) - \gamma(X_{z}, z, t, T)) dN_{t}^{r}
\]

The optional projection of \( H_{t} \) on \( \mathcal{G}_{t} \) is built as in Proposition 2.3. To conclude note that \( (N_{t}) \) is \( \mathcal{G} \)-adapted and is therefore equal to its optional projection \( \mathbb{E} (N_{t} | \mathcal{G}_{t}) \).

6.6 Proof of Corollary 2.9

Let \( Z_{N_{r}^{-}} = z \) be the last date of rating disclosure before \( t \). As \( \mathbb{E} (dN_{t}^{r} | \mathcal{G}_{t}) = \lambda H_{t} dt \) and given that \( \mathcal{M}_{t}^{Q} \) and is \( (\mathcal{G}, Q) \)-martingale, we obtain from Equation (2.6) that

\[
\mathbb{E}^{Q} (dP(t, T) | \mathcal{G}_{t}) = rP(t, T)dt + (1 - R) e^{-r(T-t)} (\mathbb{E} (\gamma(X_{t}, t, t, T) | \mathcal{G}_{t}) - \gamma(X_{z}, z, t, T)) \lambda H_{t} dt
\]

with

\[
\mathbb{E} (\gamma(X_{t}, t, t, T) | \mathcal{G}_{t}) = H_{t} \mathbb{E} (\gamma(X_{t}, t, t, T) | X_{0}, \ldots X_{z}, t < \tau)
\]

\[
= H_{t} \sum_{j=1}^{L-1} \gamma(e_{j}, t, t, T) P^{Q} (X_{t} = e_{j} | X_{z}, z < \tau)
\]

On another side, we have

\[
\gamma(e_{j}, t, t, T) = e_{j}^{T} D^{\top} \exp (B (T - t)) 1_{L-1} = P^{Q} (\tau > T | X_{t} = e_{j}, t < \tau).
\]

Equation (6.6) can then be rewritten as follows

\[
\mathbb{E} (\gamma(X_{t}, t, t, T) | \mathcal{G}_{t}) = H_{t} \sum_{j=1}^{L-1} P^{Q} (\tau > T | X_{t} = e_{j}, t < \tau) P^{Q} (X_{t} = e_{j} | X_{z}, z < \tau).
\]

Since \( (X_{t}) \) is a Markov process, we have the relation

\[
P^{Q} (\tau > T | X_{z}, z < \tau) = \sum_{j=1}^{L-1} P^{Q} (\tau > T | X_{t} = e_{j}, t < \tau) P^{Q} (X_{t} = e_{j} | X_{z}, z < \tau)
\]

that allows us to infer that

\[
\sum_{j=1}^{L-1} P^{Q} (X_{t} = e_{j} | X_{z}, z < \tau) \gamma(e_{j}, t, t, T) = \frac{P^{Q} (\tau > T | X_{z}, z < \tau)}{P^{Q} (\tau > t | X_{z}, z < \tau)} = \gamma(X_{z}, z, t, T)
\]

and that

\[
\mathbb{E} (\gamma(X_{t}, t, t, T) | X_{0}, \ldots X_{z}, t < \tau) = \gamma(X_{z}, z, t, T).
\]
We note (Rolski et al. 2009, p157). Then, if a rating has occurred, the times
in the first term of equation (6.7) becomes then

\[ C(t, S, K, T) = e^{-r(S-t)} \mathbb{E}^Q \left( H \left( e^{-r(T-S)} (R + (1-R)\gamma(X_V, V, S, T)) - K \right) \right)_{+} |X_0, ..., X_z, t < \tau \] 

First note that

\[ \mathbb{E}(H \mathbb{E}^Q |X_0, ..., X_z, t < \tau) = \gamma(X_z, z, t, S). \]

So as to lighten future calculations, we denote

\[ g(X_V, V) = \left( e^{-r(T-S)} (R + (1-R)\gamma(X_V, V, S, T)) - K \right)_{+}. \]

The expectation in the first term of equation (6.7) becomes then

\[ \mathbb{E}^Q (H \mathbb{E}^Q g(X_V, V) |X_0, ..., X_z, t < \tau) = \mathbb{E}^Q \left( \mathbb{E}^Q (X_V = e_j, \tau > S | X_z, t < \tau, V) g(e_j, V) \right) \]

(6.8)

We note \( f_V(v) \) the density of the last rating disclosure before time \( S \). If \( n \) information disclosures have occurred, the times \( Z_1, ..., Z_n \) are unordered random variables, distributed as iid uniform (see Rolski et al. 2009, p157). Then, \( f_V(v) \) that is defined for \( v \leq S \), can be written as the following sum

\[ f_V(v) = P(N_{S-t} = 0) \delta(v-t) + \sum_{k=1}^{\infty} P(N_{S-t} = k) k! (v-t)^{k-1} \frac{(S-t)^k}{(S-t)^k} \]

\[ = e^{-\lambda(S-t)} \left( \delta(v-t) + \lambda \sum_{k=1}^{\infty} \frac{(v-t)^k}{(k-1)!} \right) \]

The expectation in Equation (6.8) can then be rewritten as follows

\[ \mathbb{E}^Q \left( \mathbb{E}^Q (X_V = e_j, \tau > S | X_z, t < \tau, V) g(e_j, V) \right) \]

(6.8)

\[ = \int_{t}^{S} \mathbb{E}^Q (X_V = e_j, \tau > S | X_z, t < \tau, V = v) g(e_j, v) f_V(v) dv \]

\[ = \int_{t}^{S} \delta(X_z, e_j, z, t, v, S) g(X_V, v) f_V(v) dv. \]

6.8 Proof of Proposition 2.11

As in the proof of Proposition (2.1), we have

\[ P(t, T) = Re^{-r(T-t)} + (1-R)e^{-r(T-t)} \mathbb{E}^Q (H_T | \mathcal{G}_t) H_t, \]
By Proposition 2.11, we get that for

\[ E^Q (H_T | G_i) = P^Q \left( \tau > T \mid X_{Z_{N'_i}}, R_u \leq R_{Z_{N'_i}}, \forall u \in [Z_{N'_i}, t] \right) \]

As the numerator of this last equation is equal to

\[ P^Q(\tau > T, R_u \leq R_{Z_{N'_i}}, \forall u \in [Z_{N'_i}, t]) | X_{Z_{N'_i}}, Z_{N'_i} < \tau) \]

To simplify further calculations, let us denote:

\[ X_{Z_{N'_i}}^T D_{R_{Z_{N'_i}}}^\top \exp \left( B_{R_{Z_{N'_i}}}^{[R_{Z_{N'_i}}]} (t - Z_{N'_i}) \right) 1_{R_{Z_{N'_i}}} \]

we get the result.

### 6.9 Proof of Proposition 2.12

By Proposition 2.11, we get that for \( t \in (t_i, t_{i+1}) \)

\[
dP(t, T) = rP(t, T)dt + (1 - R) e^{-r(T-t)} \frac{\partial}{\partial t} \bar{\gamma}(X_{Z_{N'_i}}, Z_{N'_i}, t, T) H_t dt \\
+ (1 - R) e^{-r(T-t)} \tilde{\gamma}(X_{Z_{N'_i}}, Z_{N'_i}, t, T) dH_t \\
+ (1 - R) e^{-r(T-t)} \left( \bar{\gamma}(X, t, t, T) - \tilde{\gamma}(X_{Z_{N'_i}}, Z_{N'_i}, t, T) \right) dN^r_t \quad (6.9) \]

To simplify further calculations, let us denote:

\[ G^0 = D_{R_{Z_{N'_i}}}^\top \exp \left( B_{R_{Z_{N'_i}}}^{[R_{Z_{N'_i}}]} (t - Z_{N'_i}) \right) , \]

\[ G^1 = \left( B_{R_{Z_{N'_i}}}^{[R_{Z_{N'_i}}]} D_{R_{Z_{N'_i}}} | D_{R_{Z_{N'_i}}}^L 1_{R_{Z_{N'_i}}} D^\top \right) \]

First, note that, according to the definition of \( \bar{\gamma} \), its derivative is equal to

\[
\frac{\partial}{\partial t} \bar{\gamma}(X_{Z_{N'_i}}, Z_{N'_i}, t, T) = \frac{1}{X_{Z_{N'_i}}^T G^0 1_{R_{Z_{N'_i}}} \left[ -\bar{\gamma}(X_{Z_{N'_i}}, Z_{N'_i}, t, T) X_{Z_{N'_i}}^T G^0 B_{R_{Z_{N'_i}}}^{[R_{Z_{N'_i}}]} 1_{R_{Z_{N'_i}}} \\
+ X_{Z_{N'_i}}^T G^0 G^1 \exp (B (T - t)) 1_{R_{Z_{N'_i}}} \right] . \]

Second, as in the proof of Proposition 2.3, we can then infer that

\[ M_i^0 = E^Q \left( H_i^0 | G_i \right) - \int_0^t E^Q \left( a_{R_{u+L}} H_u | G_u \right) du = H_i^0 - \int_0^t E^Q \left( a_{R_{u+L}} | G_u \right) H_u du. \]
is a \((G, Q)\) martingale, where \(H_t^0 = 1_{\{\tau \leq t\}}\). Moreover we have that

\[
E^Q \left( a_{R_t, L} \mid \mathcal{G}_t \right) = \sum_{j=1}^{R_{Z_{N_t}^j}} a_{j, L} P^Q \left( X_t = e_j \mid X_{Z_{N_t}^j}^j, R_u \leq R_{Z_{N_t}^j}, \forall u \in [Z_{N_t}^j, t] \right) H_t
\]

\[
= \frac{X_{Z_{N_t}^j}^T G^0 D_{[R_{Z_{N_t}^j}]} \left( \sum_{j=1}^{R_{Z_{N_t}^j}} a_{j, L} e_j \right)}{X_{Z_{N_t}^j}^T G^0 1_{R_{Z_{N_t}^j}}^j} H_t.
\]

We can rewrite (6.9) as follows

\[
dP(t, T) = rP(t, T)dt - (1 - R)e^{-r(T-t)} \tilde{\gamma}(X_{Z_{N_t}^j}, Z_{N_t}^j, t, T) dM_t^0
\]

\[
+ (1 - R)e^{-r(T-t)} \left( \tilde{\gamma}(X_t, t, t) - \tilde{\gamma}(X_{Z_{N_t}^j}, Z_{N_t}^j, t, T) \right) dN_t^\gamma
\]

\[
- (1 - R)e^{-r(T-t)} \frac{1}{X_{Z_{N_t}^j}^T G^0 1_{R_{Z_{N_t}^j}}^j} \times
\]

\[
\left( X_{Z_{N_t}^j}^T G^0 B^{[R_{Z_{N_t}^j}]} 1_{R_{Z_{N_t}^j}} + X_{Z_{N_t}^j}^T G^0 D_{[R_{Z_{N_t}^j}]} \left( \sum_{j=1}^{R_{Z_{N_t}^j}} a_{j, L} e_j \right) \right) H_t dt
\]

\[
- (1 - R)e^{-r(T-t)} \frac{X_{Z_{N_t}^j}^T G^0 G^1 \exp \left( B (T - t) \right) 1_{L-1}}{X_{Z_{N_t}^j}^T G^0 1_{R_{Z_{N_t}^j}}} H_t dt. \tag{6.10}
\]

Now, we note that

\[
X_{Z_{N_t}^j}^T G^0 B^{[R_{Z_{N_t}^j}]} 1_{R_{Z_{N_t}^j}} + X_{Z_{N_t}^j}^T G^0 D_{[R_{Z_{N_t}^j}]} \left( \sum_{j=1}^{R_{Z_{N_t}^j}} a_{j, L} e_j \right)
\]

\[
= X_{Z_{N_t}^j}^T G^0 \left( B^{[R_{Z_{N_t}^j}]} 1_{R_{Z_{N_t}^j}} + D_{[R_{Z_{N_t}^j}]} \left( \sum_{j=1}^{R_{Z_{N_t}^j}} a_{j, L} e_j \right) \right)
\]

\[
= X_{Z_{N_t}^j}^T D^\top \exp \left( B (t - Z_{N_t}^j) \right) D_{[R_{Z_{N_t}^j}]} \left( - \sum_{j=1}^{R_{Z_{N_t}^j}} a_{j, L} \sum_{l=R_{Z_{N_t}^j}+1}^{L-1} e_j \right), \tag{6.11}
\]

and that

\[
E^Q \left( \sum_{j>R_{Z_{N_t}^j}} a_{R_t, j} H_t dt \mid \mathcal{G}_t \right) = \sum_{j=1}^{R_{Z_{N_t}^j}} \left( \sum_{l=R_{Z_{N_t}^j}+1}^{L-1} a_{j, l} \right) \frac{X_{Z_{N_t}^j}^T G^0 D_{[R_{Z_{N_t}^j}]} e_j}{X_{Z_{N_t}^j}^T G^0 1_{R_{Z_{N_t}^j}}} H_t dt
\]

\[
= \frac{X_{Z_{N_t}^j}^T G^0 D_{[R_{Z_{N_t}^j}]} \sum_{j=1}^{R_{Z_{N_t}^j}} \left( \sum_{l=R_{Z_{N_t}^j}+1}^{L-1} a_{j, l} e_j \right)}{X_{Z_{N_t}^j}^T G^0 1_{R_{Z_{N_t}^j}}} H_t dt. \tag{6.12}
\]
Moreover, we can prove the following relations,

\[ \mathbb{E}^Q \left( \tilde{\gamma}(X_t, t, T) \sum_{j > R_{Z_{N_t}^L}} a_{R_{t,j}} H_t \, | \mathcal{G}_t \right) \]

(6.13)

\[ = \mathbb{E}^Q \left( X_t^T D^T \exp \left( B(T-t) \right) 1_{L-1} \sum_{j > R_{Z_{N_t}^L}} a_{R_{t,j}} H_t \, | \mathcal{G}_t \right) \]

\[ = \sum_{l=1}^{R_{Z_{N_t}^L}} \sum_{j = R_{Z_{N_t}^L} + 1}^{L-1} a_{l,j} e_j^T D^T \exp \left( B(T-t) \right) 1_{L-1} \frac{X_{Z_{N_t}^L}^T G^0 D_{|R_{Z_{N_t}^L}^L} e_l}{X_{Z_{N_t}^L}^T G^0 1_{R_{Z_{N_t}^L}}^T} H_t \, dt \]

\[ = \frac{X_{Z_{N_t}^L}^T G^0 D_{|R_{Z_{N_t}^L}^L} \left( \sum_{j=1}^{R_{Z_{N_t}^L}} \left( \sum_{l=R_{Z_{N_t}^L} + 1}^{L-1} a_{j,l} e_l \right) e_j^T D^T \right) \exp \left( B(T-t) \right) 1_{L-1}}{X_{Z_{N_t}^L}^T G^0 1_{R_{Z_{N_t}^L}}^T} \]

and

\[ X_{Z_{N_t}^L}^T G^0 G^1 \exp \left( B(T-t) \right) 1_{L-1} = \]

\[ X_{Z_{N_t}^L}^T G^0 D_{|R_{Z_{N_t}^L}^L} \left( \sum_{j=1}^{R_{Z_{N_t}^L}} \left( \sum_{l=R_{Z_{N_t}^L} + 1}^{L-1} a_{j,l} e_l \right) e_j^T D^T \right) \exp \left( B(T-t) \right) 1_{L-1}. \]  

(6.14)

If we insert (6.11) (6.12) (6.13) (6.14) in (6.10), we can infer the existence a \((\mathcal{G}, Q)\) martingale \((\tilde{M}_t^0)\) such that

\[ dP(t, T) = rP(t, T) dt - (1 - R) e^{-r(T-t)} \tilde{\gamma}(X_{Z_{N_t}^L}, Z_{N_t}^L, t, T) d\tilde{M}_t^0 \]

\[ + (1 - R) e^{-r(T-t)} \left( \tilde{\gamma}(X_t, t, T, T) - \tilde{\gamma}(X_{Z_{N_t}^L}, Z_{N_t}^L, t, T) \right) \left( dN_t^L - \sum_{j > R_{Z_{N_t}^L}} a_{R_{t,j}} H_t \right). \]

### 6.10 Proof of Proposition 2.13

According to the proof of Proposition (2.5) (see Equation (6.4)), the price of the call can be rewritten as follows

\[ C(t, S, K, T) \]

\[ = e^{-r(S-t)} \mathbb{E}^Q \left( H_S \left( \sum_{j=R_{Z_{N_t}^L}}^{L-1} a_{R_{t,j}} (T-t) \tilde{\gamma}(X_{Z_{N_t}^L}, Z_{N_t}^L, t, T) - K \right) \right) + |\mathcal{G}_t\right) H_t \]

\[ + (1 - \mathbb{E}^Q (H_S | \mathcal{G}_t) H_t) e^{-r(S-t)} \left( e^{-r(T-S)} R - K \right)_+ \]

First note that

\[ \mathbb{E} (H_S | \mathcal{G}_t) = \tilde{\gamma}(X_{z}, z, t, S). \]

So as to lighten future calculations, we use the notation

\[ g(X_{1:t}) = \left( e^{-r(T-t)} (R + (1 - R) \tilde{\gamma}(X_{1:t}, S, T)) - K \right)_+. \]
The expectation in the first term of Equation (6.4) becomes then
\[
\mathbb{E}^Q \left( H_{S} \left[ \left( e^{-r(T-S)} (R + (1 - R)\gamma(X_S, S, S, T)) - K \right)_{+} \right] | \mathcal{G}_t \right)
\]
\[
= \mathbb{E}^Q \left( 1_{\{X_S \neq e_t\}} g(X_S) | \mathcal{G}_t \right) = \sum_{j=1}^{L-1} g(e_j)^{P^Q (X_j = e_j | \mathcal{G}_t)}
\]
\[
= \sum_{j=1}^{L-1} g(e_j) \sum_{l \leq R_{x_{n'_l}}} P^Q (X_j = e_j | X_l = e_l) P^Q (X_l = e_l | \mathcal{G}_t).
\]
\[
= \sum_{j=1}^{L-1} g(e_j) \sum_{l \leq R_{x_{n'_l}}} e_l^\top \exp(A (S - t)) e_j \frac{X_{l_{n'_l}}^{\top} D_{l_{n'_l}}^{\top} \exp \left( B_{R_{x_{n'_l}}}^{(2)} (t - Z_{n'_l}) \right) D_{R_{x_{n'_l}}^{(2)}} e_j}{X_{l_{n'_l}}^{\top} D_{l_{n'_l}}^{\top} \exp \left( B_{R_{x_{n'_l}}}^{(2)} (t - Z_{n'_l}) \right) 1_{R_{x_{n'_l}}}^{(2)}}.
\]

6.11 Proof of Proposition 3.1

We use the result of Assaf et al. (1983) that provides an analytical formula of the joint probability of survival of both issuers and the Bayes’ rule to prove that

\[
P^Q \left( \tau^Y > T | X_{t_0}, ..., X_{t_1}, \tau^Z > t, \tau^Y > t \right)
\]
\[
= P^Q \left( \tau^Y > T, \tau^Z > t | X_{t_0}, X_{t_1}, \tau^Y > t \right)
\]
\[
= P^Q \left( \tau^Z > t, \tau^Y > t | X_{t_0}, \tau^Y > t \right)
\]
\[
= \frac{X_{t_0}^{\top} D_{t_0}^{\top} \exp(B(t - t_0)) G^{(2)} \exp(B(T - t)) G^{(1)} 1_{L - 1}}{X_{t_0}^{\top} D_{t_0}^{\top} \exp(B(t - t_0)) G^{(2)} G^{(1)} 1_{L - 1}}.
\]

If \( t_1 \leq \tau^Z = t_2 \leq t \)

\[
P^Q \left( \tau^Y > T | X_{t_0}, ..., X_{t_3}, \tau^Z = t_2 \leq t, \tau^Y > t \right)
\]
\[
= P^Q \left( \tau^Y > T, X_{t_3} \in \Theta^{ZD} | X_{t_3}, t_2, \tau^Y > t \right)
\]
\[
= \frac{P^Q \left( \tau^Y > T, X_{t_3} \in \Theta^{ZD} | X_{t_3} \right)}{P^Q \left( X_{t_3} \in \Theta^{ZD}, \tau^Y > t | X_{t_3} \right)}
\]

We can conclude by noticing that

\[
P^Q \left( \tau^Y > T, X_{t_3} \in \Theta^{ZD} | X_{t_3}, t_2, \tau^Y > t_3, \tau^Z > t_3 \right)
\]
\[
= \sum_{e_j \in \Theta^{ZD}} P^Q (X_{t_3} = e_j | X_{t_3}) P^Q (\tau^Y > T | X_{t_3} = e_j)
\]
\[
= \sum_{e_j \in \Theta^{ZD}} X_{t_3}^{\top} \exp(A(t_2 - t_3)) e_j e_j^{\top} \left( D^{(3)} \right)^{\top} \exp \left( B^{(3, 3)} (T - t_2) \right) 1_{L - 1}
\]

and that

\[
P^Q \left( X_{t_3} \in \Theta^{ZD}, \tau^Y > t | X_{t_3}, \tau^Y > t_3, \tau^Z > t_3 \right)
\]
\[
= \sum_{e_j \in \Theta^{ZD}} P^Q (X_{t_3} = e_j | X_{t_3}) P^Q \left( \tau^Y > t | X_{t_3} = e_j \right)
\]
\[
= \sum_{e_j \in \Theta^{ZD}} X_{t_3}^{\top} \exp(A(t_2 - t_3)) e_j e_j^{\top} \left( D^{(3)} \right)^{\top} \exp \left( B^{(3, 3)} (t - t_2) \right) 1_{L - 1}.
\]
6.12 Proof of Proposition 3.2

First note that

\[ \text{Cov}(1_{\{\tau^Y < T\}}, 1_{\{\tau^Z > t, \tau^Y > t\}}) \]
\[ = \text{Cov}(1_{\{\tau^Y > T\}}, 1_{\{\tau^Z > t, \tau^Y > t\}}) \]
\[ = P^Q(\tau^Y > T, \tau^Z > T | X_{t_i}, \tau^Z > t, \tau^Y > t) \]
\[ - P^Q(\tau^Y > T | X_{t_i}, \tau^Z > t, \tau^Y > t) P^Q(\tau^Z > T | X_{t_i}, \tau^Z > t, \tau^Y > t) \]

By using the result of Assaf et al. (1983)

\[ P^Q(\tau^Y > T, \tau^Z > T | X_{t_i}) \]
\[ = \frac{P^Q(\tau^Y > T | X_{t_i})}{P^Q(\tau^Z > t, \tau^Y > t | X_{t_i})} \]
\[ = \frac{X_i^t D_i^T \exp(B(T - t_i)) G^{(1)} G^{(2)} 1_{L^2 - 1}}{X_i^t D_i^T \exp(B(t - t_i)) G^{(1)} G^{(2)} 1_{L^2 - 1}} \]
\[ = \gamma^Y, Z. \]

Finally, we have that

\[ P^Q(\tau^Y > T | X_{t_i}, \tau^Z > t, \tau^Y > t) = \gamma^Y(X_{t_i}, t_i, t, T) \]
\[ P^Q(\tau^Z > T | X_{t_i}, \tau^Z > t, \tau^Y > t) = \gamma^Z(X_{t_i}, t_i, t, T) \]

and

\[ \text{Var}(1_{\{\tau^Y < T\}} | X_{t_i}, \tau^Z > t, \tau^Y > t) = P^Q(\tau^Y < T | X_{t_i}, \tau^Z > t, \tau^Y > t) \]
\[ \times (1 - P^Q(\tau^Y < T | X_{t_i}, \tau^Z > t, \tau^Y > t)) \]
\[ = \gamma^Y(X_{t_i}, t_i, t, T) (1 - \gamma^Y(X_{t_i}, t_i, t, T)). \]

6.13 Proof for Equation 3.5

The relation (3.5) is the expectation of the option payoff. The probability of being in state \( i \) conditionally to the available information at time \( t \) can be developed as follows:

\[ P^Q(X_{t_S} = e_i | X_{t_i}, \tau^Z > t, \tau^Y > t) = \frac{P^Q(X_{t_S} = e_i | X_{t_i})}{\sum_{e_k \in \Theta} P^Q(X_t = e_k | X_{t_i})} \]
\[ = \frac{X_i^t e_j \exp(A(t_S - t_i)) e_j}{\sum_{e_k \in \Theta} X_i^t \exp(A(t - t_i)) e_k}. \]

And the density of the time of default of \( Z \), conditionally to the fact that \( Z \) has gone to bankruptcy between time is equal to the following ratio:

\[ f_{\tau_i | \tau_z \leq t_S}(u) = \frac{f_{t < \tau_z}(u)}{F_{t < \tau_z}(t_S)} \quad u \in [t, t_S]. \]

where \( F_{t < \tau_z}(u) \) is the cumulative distribution of the survival time of \( Z \),

\[ F_{t < \tau_z}(u) = 1 - P^Q(\tau^Z > u | X_{t_i}, \tau^Z > t, \tau^Y > t) \]
\[ = 1 - P^Q(\tau^Y > t, \tau^Z > u | X_{t_i}) P^Q(\tau^Y > t, \tau^Z > u | X_{t_i}) \]
\[ = 1 - \frac{X_i^t D_i^T \exp(B(t - t_i)) G^{(1)} \exp(B(u - t)) G^{(2)} 1_{L^2 - 1}}{X_i^t D_i^T \exp(B(t - t_i)) G^{(1)} G^{(2)} 1_{L^2 - 1}} \]

and where \( f_{t < \tau_y}(u) \) is obtained by differentiation:

\[ f_{t < \tau_z}(u) = - \frac{X_i^t D_i^T \exp(B(t - t_i)) G^{(1)} \exp(B(u - t)) B G^{(2)} 1_{L^2 - 1}}{X_i^t D_i^T \exp(B(t - t_i)) G^{(1)} G^{(2)} 1_{L^2 - 1}}. \]
References


