Systemic tail risk distribution

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2015.7
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September 27, 2015

Abstract

We introduce the systemic tail risk distribution of a financial market to characterize the asset return linkages during financial crisis. This distribution provides the probabilities that several assets of a market lose a large part of their nominal value given that the price of one of them at least collapses. It introduces a new way for assessing the stability of a financial market during potential systemic risk events. We propose a new type of multivariate extreme value distributions for high dimensional vectors to model the extremal dependence between asset prices and we use efficient likelihood inference methods to estimate the parameters of the systemic tail risk distribution. Our real data show that this distribution may be U-shaped, illustrating potential stock market crashes.

Keywords: Systemic risk; Financial crises; Heavy tails; Multivariate extreme value Theory

1 Introduction

Systemic risk refers to the risk of collapse of an entire financial market (or of an entire financial system), as opposed to the risk associated with any one individual entity, group or component of the market. It is considered as a consequence of interlinkages and interdependencies such that the failure of a single entity or cluster of entities can cause a cascading failure, which could potentially bankrupt or bring down the entire market. A systemic risk events has also the potential to have serious negative consequences on the real economy, and the recent financial crisis has clearly demonstrated the need for the financial industry, policymakers and regulators to develop a better understanding of systemic risk.

There exists a very large number of systemic risk measures that have been put forward during the last decade. They can be distinguished between indicators focusing on institutions (e.g. banks and insurance companies), indicators measuring systemic risk in financial markets and infrastructures, indicators of interconnectedness and networks, and comprehensive or system-wide indicators (see de Bandt et al. [5] for an extensive review of the different indicators available in the economic and financial literature).

A large subset of the literature on systemic risk measures for institutions relies on the information included in market prices using usual risk measure as VaR – Value at Risk – and Expected Shortfall associated to small (but not extreme) levels of probabilities, typically 5%. Four important examples of such measures are the Marginal Expected Shortfall (MES) and the Systemic Expected Shortfall (SES) of Acharya et al. [2], the Systemic Risk Measure (SRIK) of Acharya, Engle and Richardson [1] and Brownlees and Engle [11], and the Delta Conditional Value-at-Risk ($\Delta$CoVaR)
of Adrian and Brunnermeier [3]. These measures are used to distinguish between two different concepts: “Systemic importance” that refers to the impact of a firm’s default on the system and that can be estimated e.g. by the $\Delta$CoVaR, and “Systemic fragility” that is related to the firm’s vulnerability to a systemic event and that may be appraised by the MES and the SRISK measure. The systemic risk rankings of financial institutions depend highly upon the adopted measure. From a microprudential point of view, it is useful to assess individual institutions’ exposures to systemic risk by computing their systemic fragility. From a macroprudential perspective, it is more relevant to measure each institution’s systemic importance in a way to prevent systemic spillover. In a recent paper, Kupiec and Guntay [29] have shown by performing extensive simulations that these systemic risk measures may be subject to model errors and that (non parametric) $\Delta$CoVaR and MES statistics are in particular unlikely to detect asymptotic tail dependence unless the tail dependence is strong.

Another important set of contributions in the literature on systemic risk measures for institutions relies on default probability. Instead of measuring potential losses of financial institutions under an systemic event it focuses on quantifying the likelihood of the failure of a financial institution by considering the default probabilities (Gray and Jobst [18], Huang et al. [25] – Distress Insurance Premium, Segoviano and Goodhart [36] – Banking Stability Measures). A final strand of the literature relies on the time series Granger-causality which is interpreted as a spill-over effect (see Billio et al. [9]).

Systemic risk should be considered as a tail risk, but strangely the idea to implement extreme value theory for measuring systemic risk on market prices has not be looked deeper. Exceptions are given in a list of papers by Hartmann, Straetmans and de Vries ([22], [23] and [24]). In particular in [23], they propose indicators of the severity and structure of system risk in the U.S. and European banking systems from asymptotic interdependencies between banks’ equity prices and they use tools available from bivariate extreme value theory to estimate individual banks’ exposure to each other and to systematic risk. The present paper adds a new perspective to the systemic risk and to the linkages between assets in a financial market by using a novel methodology based on multivariate extreme value distributions in high dimensions. Its aim is to measure the degree of (instantaneous) contagion in a stressed financial market without putting more importance on some specific firms. More specifically we intend to study the distribution of the number of assets that simultaneous crash down given a collapse of at least one asset. We do not favor some specific firms, but rather look at the dependencies between asset prices to measure the contagion effect. This approach introduces a new way for assessing the stability of a financial market during potential systemic risk events.

Models and inference methods for high dimensional extreme value distributions are however research fields that are in their beginning of development. In Bienvenu and Robert [8], we provided new full likelihood methods for vectors of exceedances that are numerically strongly efficient. We also introduced a new type of multivariate extreme value distributions for high dimensional vectors that we call the homogeneous “clustered” maxstable distributions and we propose to use it to identify and to fit extreme behaviors of financial returns. To illustrate our approach, we consider in the paper a model where asset price dynamics are given by a CAPM model whose market return is driven by a GARCH process and where beta coefficients are stochastic. In this framework the tail dependence function of the asset returns is linked to the distribution of the vector of beta coefficients. Our real data application will show that the family of homogeneous “clustered” maxstable distributions is suitable to model extreme dependencies and that it can be assumed that the vector of betas has an exchangeable distribution. We then derive the tail risk distribution and notice that this distribution is U-shaped, illustrating potential stock market crashes.
The paper is organized as follows. Section 2 gives the definition of the tail risk distribution for a financial market, characterizes this distribution with multivariate extreme value theory and introduces the CAPM model and the family of homogeneous “clustered” maxstable distributions. In Section 3 we present our likelihood inference method. Section 4 gives a real data application on financial data. Proofs have been deferred to the appendix.

2 Systemic tail risk distribution

2.1 Definition

We begin by the formal definition of the systemic tail risk distribution. We consider \( m \) firms and denote by \( r_{jt} \) the return of firm \( j \) at day \( t \) and by \( q_{jt}(\cdot) \) its respective quantile function. We are interested by the distribution of the number of firms whose returns are smaller than their quantile associated with the level of probability \( \eta \) given that there is at least one firm whose return is smaller than its own quantile. Let us denote by

\[
N_{mt}(\eta) = \sum_{j=1}^{m} I\{r_{jt} \leq q_{jt}(\eta)\}
\]

this number. The limiting distribution \( (P_j)_{j=1,...,m} \) defined by

\[
P_j = \lim_{\eta \to 0} \Pr(N_{mt}(\eta) = j | N_{mt}(\eta) \geq 1), \quad j = 1, \ldots, m,
\]

is called the systemic tail risk distribution and measures the (instantaneous) contagion effect in the market.

Hartman et al. ([22], [23] and [24]) considered the case where \( m = 2 \) and defined as a linkage indicator the expectation of this distribution, i.e. \( P_1 + 2P_2 \). In our model, \( m \) can be very large (e.g. 100) and systemic risk arises now in the tail of this distribution. We are in particular interested in seeing whether such a distribution may be U-shaped, illustrating potential stock market contagions and crashes.

Let us denote by \( Y_{jt} = 1/F_{jt}(r_{jt}) \) the reciprocal of the ‘uniform’ return of firm \( j \) where \( F_{jt}(\cdot) \) is the probability distribution function of \( r_{jt} \). Small values of \( r_{jt} \) give large values for \( Y_{jt} \). Since \( N_{mt}(\eta) = \sum_{j=1}^{m} I\{Y_{jt} \geq \eta^{-1}\} \), we now focus on extreme (positive) values of \( Y_t = (Y_{1t}, \ldots, Y_{mt}) \).

2.2 Characterization of the systemic tail risk distribution with multivariate extreme value theory

Let us denote by \( Y \) a random vector with the same distribution as \( Y_t \). We assume that \( Y \) belongs to the maximum domain of attraction of a multivariate extreme value distribution \( G_* \). Then there exists a random vector \( U = (U_1, \ldots, U_m) \) on \( \mathbb{R}^m \) with \( \mathbb{E}[U_j^+] = 1 \) for \( j = 1, \ldots, m \) where \( U_j^+ = \max(0, U_j) \), such that, as \( T \) tends to infinity,

\[
[\Pr(Y \leq Tz)]^T \to G_*(z) = \exp(-V_*(z))
\]

with \( \{Y \leq Tz\} = \cap_{j=1,...,m} \{Y_j \leq Tz_j\} \) and

\[
V_*(z) = \mathbb{E} \left[ \max(z_1^{-1}U_1^+, \ldots, z_m^{-1}U_m^+) \right]
\]
(see Corollary 9.4.5 and Remark 9.4.8 in de Haan and Ferreira [20] and Corollary 2.7 in Oesting et al. [33]). Moreover it may be shown that, if $Z$ is a random vector with distribution $G_*$, then
\[ Z \overset{d}{=} \max_{j \geq 1} \zeta_j U_j, \]
where $(\zeta_j)_{j \geq 1}$ are the points of a Poisson process on $\mathbb{R}^+$ with intensity $ds/s^2$, and $(U_j)_{j \geq 1}$ is a sequence of independent and identically distributed random vectors with the same distribution as $U$. The random vector $U$ is referred to as the spectral random vector of $G_*$ (or sometimes as the profile vector). It should be underlined that the multivariate extreme value distribution $G_*$ is not defined by a unique spectral random vector $U$. Indeed, if $A$ is a positive random variable independent of $U$ and with a unit mean, then
\[ E\left[ \max_{z^1 \cdots z^m} z^1 U_1^1 + \cdots + z^m U_m^m \right] = V_*(z). \]

We easily derive from the condition that $Y$ belongs to the maximum domain of attraction of $G_*$ an asymptotic equivalent form of the probability of the systemic risk event. Indeed, since
\[ \{N_{mt}(\eta) \geq 1\} = \bigcup_{j=1}^{m} \{Y_j > \eta^1\}, \]
we see that
\[ \lim_{\eta \to 0} \eta^{-1} \Pr(N_{mt}(\eta) \geq 1) = \lim_{T \to \infty} -T \log \Pr(Y \leq T e) = -\log G_*(e) = E[\max_{j=1,\ldots,m} U_j]. \]

The next proposition establishes that the systemic tail risk distribution is also characterized by $U$. We first introduce some notation. Let $\mathcal{P} = \mathcal{P}(I)$ be the power set of $I = \{1, \ldots, m\}$ without $\emptyset$ and define, for $B \in \mathcal{P}$,
\[ V_B^* = \int_0^\infty P(U_B > \gamma e_B, U_{B^c} \leq \gamma e_{B^c}) d\gamma \]
with $\{U_B > \gamma e_B\} = \cap_{j \in B} \{U_j > \gamma\}$. Moreover, let $p_B = V_B^*/V^*$ where $V^* = V^*(e)$. Since
\[ V^* = \int_0^\infty \Pr\left( \max_{j=1,\ldots,m} U_j > \gamma \right) d\gamma = \int_0^\infty \Pr(\cup_{j=1}^{m} \{U_j > \gamma\}) d\gamma, \]
we deduce that $\sum_{B \in \mathcal{P}} V_B^* = V^*$ and that $(p_B)_{B \in \mathcal{P}}$ is a discrete probability distribution.

**Proposition 1** Assume that $Y$ belongs to the maximum domain of attraction of a multivariate extreme value distribution with a spectral random vector $U$. Then
\[ P_j = \lim_{\eta \to 0} \Pr(N_{mt}(\eta) = j | N_{mt}(\eta) \geq 1) = \sum_{B \in \mathcal{P}, |B| = j} p_B(e). \]

The proof is given in Appendix 1. Note that it could also be possible to characterize the asymptotic probabilities that some specific firms crash down given that one crashes down.
2.3 The spectral random vector for a CAPM model with stochastic betas

The Capital Asset Pricing Model (CAPM) due to Sharpe [37] and Lintner [31] relates the expected return of an asset to the expected return of the market portfolio. A key parameter of this model is the beta coefficient which is the slope of the market model:

\[ r_{jt} - r_f = \beta_j (r_{Mt} - r_f) + \varepsilon_{jt}, \quad j = 1, \ldots, m, \quad t = 1, \ldots, T, \]

where \( r_{jt} \) and \( r_{Mt} \) are the returns of asset \( j \) and of the market portfolio at time \( t \) respectively, \( r_f \) is the risk free rate of the market, and \( \varepsilon_{jt} \) is the idiosyncratic risk or diversifiable risk of asset \( j \). The beta is thus interpreted as the sensitivity of the asset return to changes in the return of the market portfolio. Commonly, the beta coefficients are estimated by Ordinary Least Squares applied to the linear regressions for each asset. A lot of empirical studies have shown that the expected returns of assets cannot be perfectly explained by the market factor because betas are not stable over time and should rather be considered as stochastic to predict or explain the cross-section of asset returns (see e.g. the conditional CAPM of Hansen and Richard [21]). There is no agreement about the best process assumption for beta dynamics. Stochastic models for betas assume in general that betas have a constant mean and a random component, or that betas have a Gaussian mean reverting dynamics (conditional or not) to observed instruments. Beta estimates are essential for many areas of modern finance, including asset pricing, cash flow valuation or performance evaluation.

We will see that it is also the case for risk management of extreme events.

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are used to model volatility of the market portfolio and heavy tailedness of its returns (Engle [15], Bollerslev [10]). GARCH processes have indeed power law marginal tails and, more generally, regularly varying finite-dimensional distributions (see e.g. Davis and Mikosch [14]). Let us assume that the returns of the market portfolio follows a GARCH process such that

\[ \Pr(r_{Mt} > r) \sim c_{M+} r^{-\kappa}, \quad \Pr(r_{Mt} < -r) \sim c_{M-} r^{-\kappa}, \quad r \to \infty, \]

with \( c_{M+} > 0, c_{M-} > 0 \) and \( \kappa > 0 \). We then consider a financial model with asymmetric beta in bull and bear market conditions

\[ r_{jt} = \alpha_i + \beta_j^+ r_{Mt}^+ \mathbb{I}_{r_{Mt} > 0} + \beta_j^- r_{Mt}^- \mathbb{I}_{r_{Mt} \leq 0} + \varepsilon_{jt}, \quad j = 1, \ldots, m, \quad t = 1, \ldots, T, \]

where \((\beta_j^+)_{j=1,\ldots,m}\) and \((\beta_j^-)_{j=1,\ldots,m}\) are independent of \( r_{Mt} \) and \( \varepsilon_{jt} \) for \( t = 1, \ldots, T \) and such that

\[ \mathbb{E}[(\beta_j^+)\kappa] < \infty \text{ and } \mathbb{E}[(\beta_j^-)\kappa] < \infty \text{ for all } j, \ldots, m. \]

Moreover we assume that they have lower (positive and negative) tails than the marginal distribution of the market portfolio.

Using Breiman’s lemma, it may be shown that \((r_{1t}, \ldots, r_{mt})\) satisfies

\[ \lim_{r \to -\infty} r \Pr \left( \bigcup_{j=1}^{m} \{r_{jt} < q_{jt}(r^{-1})z_j^{1/\kappa} \} \right) = \lim_{T \to \infty} T \Pr \left( \bigcup_{j=1}^{m} \{Y_{jt} > Tz_j \} \right) = \mathbb{E}\left[ \max_{j=1,\ldots,m} z_j^{-1}(\beta_j^+)\kappa / \mathbb{E}(\beta_j^+)\kappa \right]. \]

(see Appendix 2). It follows that \( U \) may be chosen as \((\beta_1^+)\kappa / \mathbb{E}(\beta_1^+)\kappa, \ldots, (\beta_m^+)\kappa / \mathbb{E}(\beta_m^+)\kappa\) and that the systemic tail risk distribution strongly depends on the dependence structure of \((\beta_1^+, \ldots, \beta_m^+)\).

2.4 The homogeneous clustered max-stable family

Let \((\pi_i)_{i=1,\ldots,f}\) be a partition of \( \mathcal{I} = \{1, \ldots, m\} \). An homogeneous clustered max-stable extreme value distribution has a spectral random vector \( U \) such that \( U = (U_{\pi_1}, \ldots, U_{\pi_f}) \) is a random vector in \( \mathbb{R}^m \) with \( \mathbb{E}[U^+_{j}] = 1 \) for \( j = 1, \ldots, m \), \( U_{\pi_1}, \ldots, U_{\pi_f} \) are independent sub-vectors and

\[ \Pr(U_{P_i} \leq z_{P_i}) = C_i \left( (F_i(z_j))_{j \in \pi_i} \right) \]
where $C_i$ is copula function and $F_i$ is the common marginal distribution function of $U_{P_i}$. The distribution is called homogeneous clustered because there is a unique marginal distribution function for each block of the partition.

When $(\pi_i)_{i=1,...,I}$ is the trivial partition, and $C$ is the independent copula, it is possible to derive several well-known analytical forms for the tail-dependence functions (e.g. the Marshall-Olkin distribution \[32\], the Galambos distribution \[17\], the Logistic (or the Gumbel) distribution \[19\], see Beinvenüe and Robert \[8\]). Let $U_j$, for $j = 1, ..., m$, be positive independent and identically distributed random variables with distribution function $H$ and unit mean, simple calculations give

$$V_*(z) = \sum_{l=1}^{m} z_l^{-1} \mathbb{E} \left[ U \prod_{s \neq l} H_{U} \left( \left( \frac{z_s}{z_l} \right) U \right) \right]$$

where $U$ is a positive random variable with distribution $H_U$.

If the copulas $C_i$ are Gaussian and the distributions $F_i$ are Log-normal, then $V_*(z)$ may be written as a linear combination of $(m-1)$-dimensional multivariate normal probabilities (see Hüsler and Davison \[26\]) and the probabilities $p_B$ may be efficiently computed (see Bienvenüe and Robert \[8\]). If the copulas $C_i$ are Gaussian and the distributions $F_i$ are the distributions of powers (of the positive part) of a Gaussian random variable, then $V_*(z)$ may be written as a linear combination of $(m-1)$-dimensional multivariate Student probabilities (see e.g. Optiz \[34\]) and the probabilities $p_B$ may be also efficiently computed (see Bienvenüe and Robert \[8\]).

Finally note that the systemic tail risk distribution has a quasi-explicit analytical expression when $(\pi_i)_{i=1,...,I}$ is the trivial partition and $C$ is an Archimedean copula. Let $\psi$ be its generator, then we have

$$P_j = \sum_{k=0}^{j} \frac{(-1)^{k-1}}{j \choose k} \frac{\int_0^\infty \left[ 1 - \psi \left( (m-j+k)\psi^{-1} (F (\gamma)) \right) \right] d\gamma}{\int_0^\infty \left[ 1 - \psi \left( m\psi^{-1} (F (\gamma)) \right) \right] d\gamma}$$

(see Appendix 3).

### 3 Likelihood inference

Likelihood inference of multivariate extreme value distributions in high dimension is infeasable in general because likelihood functions are unknown. The composite likelihood methods that are based on combinations of valid likelihood objects related to small subsets of data have been rather naturally considered as an appealing way to deal with inference in high dimensions. The merit of composite likelihood is to reduce the computational complexity. These methods go back to Besag \[6\] and have been extensively studied in particular for the case where the full likelihood is replaced by quantities that combine bivariate or trivariate joint likelihoods, see e.g. Lindsay \[30\], Cox and Reid \[13\], Varin \[38\]. Recent works concerning the application for random samples in the max-domain of attraction of a max-stable distribution can be found in Bacro and Gaetan \[4\], Jeon and Smith \[27\] and Kiriliouk et al. \[28\].

In Bienvenue and Robert \[8\], we however provided quasi-explicit analytical expressions of the full likelihood of vectors of exceedances for several classes of multivariate distributions that include the homogeneous clustered max-stable family. Let us explain our approach.

We still assume that $Y$ belongs in the max-domain of attraction of $G_*$ with spectral random vector $U$, but now also assume that $U$ has an absolutely continuous distribution. For a given
threshold \( u > 0 \), we censor components that do not exceed the threshold and consider the vector \( \mathbf{X}_u = u^{-1}\mathbf{Y} \vee \mathbf{e} \) with \( \mathbf{e} = (1, \ldots, 1)' \in \mathbb{R}^m \). We are interested in the asymptotic conditional distribution of \( \mathbf{X}_u \) for other examples. In characterized in the following way:

\[
\mu (B; \mathbf{z}) = \int_0^\infty \gamma^{\|B\|} \Pr (\mathbf{U}_B \leq \mathbf{z}_B \mid \mathbf{U}_B = \mathbf{z}_B) f_{\mathbf{U}_B} (\mathbf{z}_B) \ d\gamma
\]

(3.1)

where \( B^c = \mathcal{I} \setminus B \) and \( f_{\mathbf{U}_B} \) is the probability density function of \( \mathbf{U}_B \). Note that, if the conditional distribution of \( \mathbf{U}_B \) given \( \mathbf{U}_B \) and the density functions \( f_{\mathbf{U}_B} \) are analytically known, then \( \mu (B; \mathbf{z}) \) may be easily computed because it is a one-dimensional integral. For the homogeneous clustered max-stable distribution introduced in Section 2.4 for which the copulas \( C_i \) are archimedean copulas with generators \( \psi_i \), it may be shown that

\[
\mu (B; \mathbf{z}) = \int_0^\infty \gamma^{\|B\|} \prod_{i=1}^I \left( \frac{\psi_i (\gamma)}{\psi_i (\gamma)} \right) \left( \frac{\psi_i^{-1} (F_i (\gamma z_j))}{\psi_i^{-1} (F_i (\gamma z_j))} \right) f_i (\gamma z_j) \ d\gamma
\]

(3.2)

(see Bienvenüe and Robert [8] for other examples).

For \( B \in \mathcal{P} \), let \( A_B = \{ \mathbf{x} \in [1, \infty)^m : x_i > 1, i \in B, x_i = 1, i \in B^c \} \). Note that \( \mathbf{X}_u \) takes its values in \( A = \bigcup_{B \in \mathcal{P}} A_B \).

**Proposition 2** [8] As \( u \to \infty \), \( \mathbf{X}_u \) converges in distribution to \( \mathbf{X}^* \) whose density function \( f_{\mathbf{X}^*} \) is characterized in the following way:

\[
f_{\mathbf{X}^*} (\mathbf{x}) = \frac{1}{\sum_{i=1}^m \mu \{ i \} : \mathbf{e}} \sum_{B \in \mathcal{P}} \mu (B; \mathbf{x}) \mathbb{1}_{\{ \mathbf{x} \in A_B \}}.
\]

We now impose that the probability density function of \( \mathbf{U} \) belongs to some parametric family \( \{ f_{\mathbf{U}} (\cdot; \theta) : \theta \in \Theta \} \) where \( \Theta \) is a compact set in \( \mathbb{R}^p \), for \( p \geq 1 \), and let

\[
\mu (\theta; B, \mathbf{z}) = \int_0^\infty \int_{-\infty}^{\mathbf{z}_B} \gamma^{m} f_{U_B, U_B} (u_B \gamma, \mathbf{z}_B \gamma; \theta) \ du_B \ d\gamma.
\]

We assume that:

- **C1**: \( \nabla_{\theta} \mu (\theta; B, \mathbf{z}) \) and \( \nabla^2_{\theta \theta} \mu (\theta; B, \mathbf{z}) \) exist for any \( \mathbf{z} \in \mathbb{R}^m_+ \).

- **C2**: It is possible to interchange differentiation (with respect to \( \theta \)) and integration (with respect to \( \gamma \) and \( \mathbf{z}_B \)) for

\[
\int_0^\infty \int_{-\infty}^{\mathbf{z}_B} \gamma^{m} \nabla_{\theta} f_{U_B, U_B} (u_B \gamma, \mathbf{z}_B \gamma; \theta) \ du_B \ d\gamma \quad j = 0, 1, 2.
\]

- **C3**: There exists \( \alpha > 0 \) such that, uniformly for \( B \in \mathcal{P} \) and \( \mathbf{x} = (\mathbf{x}_B, \mathbf{e}_B) \in A_B \), as \( t \to \infty \),

\[
\Pr (\mathbf{X}_u \in (\mathbf{e}, \mathbf{x})) - F_{\mathbf{X}^*} (A_B) (\mathbf{x}_B) = O (t^{-\alpha}),
\]

where \( F_{\mathbf{X}^*} (A_B) \) is the multivariate distribution function of \( \mathbf{X}^* \) restricted to \( A_B \).
Let \((Y_i)_{i=1,...,n}\) be independent and identically distributed vectors distributed as \(Y\). Let \(1 \leq k \leq n\) and for \(i = 1, \ldots, n\), we censor components that do not exceed the threshold \(n/k\) and define

\[X_{i,k} = \frac{k}{n} Y_i \vee e.\]

We only keep random vectors \(X_{i,k}\) for which \(||X_{i,k}||_\infty > 1\) and let

\[N_k = \{X_{i,k} : 1 \leq i \leq n, ||X_{i,k}||_\infty > 1\}.\]

We consider an identifiable (pseudo)-parametric statistical model \(F_k = \{\ell_{X^*}(\theta; x_i,k), \theta \in \Theta, x_{i,k} \in A, i \in N_k\}\) whose (pseudo)log-likelihood, for \(x \in A\), is given by

\[
\ell_{X^*}(\theta; x) = \sum_{B \in \mathcal{P}} \log \mu(\theta; B, (x_B, e_{B^c})) \mathbb{1}_{\{x \in A_B\}} - \log \sum_{l=1}^m \mu(\theta; \{l\}, e).
\]

Conditions C1 and C2 imply the existence of the score function \(\nabla_{\theta} \ell_{X^*}(\theta; x)\) and the Hessian function \(\nabla_{\theta}^2 \ell_{X^*}(\theta; x)\). The maximum likelihood estimator \(\hat{\theta}_k\) is then defined by the following condition

\[
\sum_{i \in N_k} \nabla_{\theta} \ell_{X^*}(\hat{\theta}_k; x_{i,k}) = 0. \tag{3.3}
\]

This estimator is asymptotically Gaussian under conditions C1, C2 and C3.

**Proposition 3** [8] Assume that C1, C2 and C3 hold. If, as \(n \to \infty\), \(k \to \infty\) such that \(k = o\left(n^{2\alpha/(1+2\alpha)}\right)\), then

\[
\sqrt{k}(\hat{\theta}_k - \theta_0) \overset{d}{\to} \mathcal{N}(0, V_{x^*}^{-1}(\theta_0)I_{X^*}^{-1}(\theta_0))
\]

where \(I_{X^*}(\theta) = \mathbb{E}[(\nabla_{\theta} \ell_{X^*}(\theta; X^*)) (\nabla_{\theta} \ell_{X^*}(\theta; X^*))^T].\)

We developed a R package, HiDimMaxStable, available on CRAN, where we implemented in particular this method.

### 4 A real data example

We now evaluate the relevance of the homogeneous “clustered” max-stable model for assessing the systemic risk within a stock market. The data are those used by Fan, Liao and Mincheva [16] to estimate high dimensional covariance matrices with a conditional sparsity structure. These data were obtained from the Center for Research in Security Prices database and consist of \(p = 100\) (anonymous) stocks with their annualized daily returns for the period January 1st, 2000, to December 31st, 2010 (approximately 2700 observations for each asset). The stocks have been chosen from several different industry sectors (more specifically, ‘consumer goods-textiles and apparel clothing’, ‘financial-credit services’, ‘healthcare-hospitals’, ‘services-restaurants’ and ‘utilities-water utilities’), with 10 stocks from each sector.

We focused on the negative returns of the assets and we transformed the marginal distributions of these negative returns of the assets such that they became approximately unit Pareto distributions by using their respective order statistics.

To fit an homogeneous “clustered” max-stable distribution, we need to choose a partition \((\pi_i)_{i=1,...,I}\) such that the components of the spectral random vector in blocks are the most dependent as possible and such that the assumption of independence between the blocks of the partition
is the most reasonable as possible. We propose to use the Partitioning Around Medoids (PAM) algorithm which has been introduced by Bernard et al. [7]. This algorithm is a specific clustering algorithm for extreme values. Let $Z$ be a random vector with distribution $G$, and spectral random vector $U$. The $F$-madogram between $Z_i$ and $Z_j$ is defined by

$$d_{ij} = \frac{1}{2} E \left[ |\exp(-Z_i^{-1}) - \exp(-Z_j^{-1})| \right] = \frac{1}{2} E \left[ \max(U_i^+,U_j^+) \right] - 1$$

(see Colley et al. [12]) and measures the dependence between the components $Z_i$ and $Z_j$ of $Z$. It can be easily evaluated with non-parametric estimators (by calculating block maxima). The PAM algorithm divides the $m$ components into $p$ clusters in the following way: (i) select randomly an initial set of $p$ medoids, (ii) form $p$ clusters by assigning every point to its closest medoid (using the distance matrix $(d_{ij})$), (iii) for each cluster, find the new medoid for which the total intra-cluster distance based on $d_{ij}$ is minimized, (iv) if at least one medoid has changed, then go back to (ii), otherwise end the algorithm.

We performed the PAM algorithm for maxima calculated over return blocks of size 50. The average silhouette widths for the entire data set suggested us to choose three clusters $P_i$ of respective sizes 20, 58 and 22. It is not possible to interpret the composition of the clusters with respect to the characteristic of the assets due to the anonymity of the assets, but it is expected that the extreme returns are mainly clustered within companies in the same industrial sectors.

We then considered for each cluster three copulas with parameter $\theta$: the Gumbel copula (with generator $\psi(t) = \exp(-t^{1/\theta})$), the Clayton copula (with generator $\psi(t) = (1 + t)^{-1/\theta}$), the Frank copula (with generator $\psi(t) = -\log(1 - e^{-t(1 - e^{-\theta})})$), and three distributions with parameter $\alpha$: the lognormal distribution (associated to the Gaussian distribution with standard error $\alpha$ and mean equal to $-\alpha^2/2$), the Weibull distribution (with shape parameter $\alpha$ and a scale parameter such that its expectation is equal to one) and the Fréchet distribution (with shape parameter $\alpha$ and a scale parameter such that its expectation is equal to one).

We estimated the pairs $(\theta, \alpha)$ for the nine combinations of copulas and distributions and for each block. We fixed $k/n = 0.1$. To choose the most appropriate pairs, we used the chi-squared distance between the empirical distribution of $N_m$ and the associated theoretical distribution using the estimated parameters. The values of the chi-squared distances for Cluster 1 are given as examples in Table 1. The Weibull distribution and the Clayton copula appear as the most suitable pair of distribution and copula for Cluster 1. Actually it is also the case for the two other clusters.

Table 1: For Cluster 1, chi-squared distances between the empirical distribution of the number of components of the normalized vector exceeding the threshold 1, and the associated theoretical distribution using the estimated parameters for the nine combinations of copulas and marginal distributions.

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<th>Weibull</th>
<th>Fréchet</th>
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<td>0.130</td>
<td>0.632</td>
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Table 2 gives for the three clusters the maximum likelihood estimates of the parameters $\theta$ and $\alpha$ for the combinations of copula and distribution that minimize the chi-squared distance. For these combinations, we re-estimated in a second step jointly all the parameters by using (3.2) for the likelihood with initial conditions given by the estimates of the pairs derived from the first step. Table 2 also provides the values of the joint maximum likelihood estimates.
Table 2: For the first step: maximum likelihood estimates of the parameters $\theta_i$ and $\alpha_i$ for each cluster $i = 1, 2, 3$ for the combination of copula and marginal distribution that minimizes the chi-squared distance between the empirical distribution of $N_m$ and the theoretical one. For the second step: jointly maximum likelihood estimates of the parameters $\theta_i$ and $\alpha_i$ based on the choices for the copula and marginal distribution of the first step. The standard errors have been computed by simulations.

We observe excellent matches between empirical and theoretical distributions of the number of exceeding components for the Cluster 1 and 2 and a very good match for Cluster 3 after the first step. This should validate the use of an homogeneous “clustered” max-stable model for these data. However the estimates of the parameters obtained after the second step differ significantly from the estimates of the parameters of the first step for Cluster 1 and 3. Moreover the chi-squared distances for these two clusters increases in an important way after the second step.

Since parameters of Cluster 1 and 3 are quite close for the first and the second steps, we decided to gather these two clusters into a single cluster 1∪3. We then estimated the parameters of the copulas and of the distributions for the two new clusters, separately as well as jointly. The values are given in Table 3.

Table 3: For the first step: maximum likelihood estimates of the parameters $\theta_i$ and $\alpha_i$ for clusters $i = 1∪3$, 2 and 1∪2∪3. For the second step: jointly maximum likelihood estimates of the parameters $\theta_i$ and $\alpha_i$ for clusters $i = 1∪3$ and 2. The standard errors have been computed by simulations.
Figure 2: Left: Empirical distributions of $N_m$ with their theoretical distributions that minimizes the chi-squared distance for the three clusters. Right: Empirical distributions of $N_m$ and their theoretical distributions with parameters derived from the second step of the estimation procedure.

The estimates of the parameters of the new cluster 1U3 for the first step are roughly the same as those obtained when it was split into two clusters 1 and 3. The estimates of the parameters obtained after the second step differ again from the estimates of the parameters of the first step and provide values that are close to the values of Cluster 2. We therefore decided to consider only
one cluster $1 \cup 2 \cup 3$ for these data. Figure 3 shows that there is a very good match between the empirical distributions of the number of exceeding components and the theoretical distribution. This is confirmed by the chi-squared distance which is small for distributions with 100 possible values. We conclude that the clustered max-stable model with only one cluster appears as a reasonable (exchangeable variable) model for these data.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{c-clayton_m—weibull}
\caption{Empirical distribution of $N_m$ with its theoretical distribution that minimizes the chi-squared distance for the entire market.}
\end{figure}

We have studied the extremal dependence between the asset prices from a static point of view, i.e. we have considered the extremal dependence of the marginal distributions of the asset prices rather than the one of their conditional distributions. It could be argued that contagion effect may be viewed as a consequence of volatility transmission between the assets. We therefore fit ARMA($p, q$)-GARCH($p', q'$) models for each asset prices (see Tables 4 and 5 in Appendix 4 for the orders $p, q, p', q'$ of each model) and use the same approach as previously but now on the studentized residuals. Figure 4 gives the empirical conditional distribution of $N_m$ with its theoretical distribution that minimizes the chi-squared distance with respect of the choice of copulas with their marginal distributions. The best copula is now given by the Frank copula. We observe that the tail of the conditional distribution is not so heavy as the one of the marginal distribution, but still lets probabilities of potential stock market crashes not negligible.
Figure 4: Empirical conditional distribution of $N_m$ with its theoretical distribution that minimizes the chi-squared distance for the entire market.

5 Conclusion

We have proposed the systemic tail risk distribution as a synthetic indicator of the systemic risk of a financial market. The tail of this distribution quantifies the risk that the market collapses. We have used extreme value theory for high dimensional vectors to characterize this distribution and have provided quasi-explicit analytical expressions of the full likelihoods for random samples in the max-domain of attraction of the homogeneous clustered max-stable distribution. Our real data application has shown that the homogeneous clustered max-stable models can be successfully applied to estimate the high-dimensional dependence structure of extreme negative returns. Results have to be confirmed for other financial markets.

Acknowledgments

We would like to thank Christian Gouriéroux, Christophe Hurlin and Nour Meddahi for fruitful discussions during the Conference “Nouveaux développements dans la modélisation et la prévision des risques extrêmes en finance”, Marseille, May 19-20, 2015.
References


6 Appendix

6.1 Appendix 1 - Proof of Proposition 1

Let us begin by a lemma.

**Lemma 4** Let \( \varphi : [0, \infty) \to \mathbb{R}^+ \) be such that \( \int_0^\infty \varphi (\gamma^{-1}) \, d\gamma < \infty \). If \( \Gamma \) is a random variable with a unit Pareto distribution, then

\[
\lim_{T \to \infty} T \mathbb{E} [\varphi (T^{-1} \Gamma)] = \int_0^\infty \varphi (\gamma^{-1}) \, d\gamma.
\]

**Proof:** We have

\[
T \mathbb{E} [\varphi (T^{-1} \Gamma)] = T \int_1^\infty \varphi (T^{-1} \gamma) \gamma^{-2} d\gamma = \int_0^T \varphi (v^{-1}) \, dv - \int_0^\infty \varphi (v^{-1}) \, dv. \quad \square
\]

Let \( \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_m) \) be defined by \( \tilde{Y}_j = \Gamma U_j \) for \( j = 1, \ldots, m \) where \( \Gamma \) is a random variable with unit Pareto distribution and independent of \( U = (U_1, \ldots, U_m) \). Then \( \tilde{Y} \) belongs to the max-domain of attraction of \( G_* \). Indeed by Lemma 4, we have

\[
\lim_{T \to \infty} T \log \Pr (\tilde{Y} \leq Tz) = - \lim_{T \to \infty} T \Pr \left( \Gamma \max_{j=1,\ldots,m} z_j^{-1} U_j > T \right)
= - \int_0^\infty \Pr \left( \max_{j=1,\ldots,m} U_j > \gamma \right) d\gamma
= - \mathbb{E} \left[ \max_{j=1,\ldots,m} z_j^{-1} U_j^+ \right].
\]

Let us now consider \( X_u = u^{-1} Y \vee e \) and note that

\[
\{ N_{mt}(\eta) = j \} = \{ \sharp \{ i : \eta Y_i \geq 1 \} = j \} = \{ \sharp \{ i : X_{m-1} > 1 \} = j \}.
\]

Since \( \tilde{Y} \) belongs to the max-domain of attraction of \( G_* \) as \( Y \), we may assume without loss of generality that \( X_u = u^{-1} \tilde{Y} \vee e \) instead of \( u^{-1} Y \vee e \) to establish our asymptotic result. Let
\[\|X_u\|_\infty = \max_{j=1,\ldots,m} X_{iu}.\] Then we have

\[
\lim_{\eta \to 0} \Pr (N_{mt}(\eta) = j | N_{mt}(\eta) \geq 1) = \lim_{u \to \infty} \Pr (\mathcal{E}\{i : X_{iu} > 1\} = j | \cup_{i=1}^{\eta} \{X_{iu} > 1\})
\]

\[
= \lim_{u \to \infty} \Pr (\cup_{B \in \mathcal{P}, |B| = j} \{X_{Bu} > e_B, X_{Bu'} = e_{Bu'}\} | \|X_u\|_\infty > 1)
\]

\[
= \sum_{B \in \mathcal{P}, |B| = j} \lim_{u \to \infty} \Pr (X_{Bu} > e_B, X_{Bu'} = e_{Bu'}) \Pr (\|X_u\|_\infty > 1)
\]

\[
= \sum_{B \in \mathcal{P}, |B| = j} \lim_{u \to \infty} \frac{\Pr (X_{Bu} > e_B, X_{Bu'} = e_{Bu'})}{\Pr (\|X_u\|_\infty > 1)}
\]

\[
= \sum_{B \in \mathcal{P}, |B| = j} \lim_{u \to \infty} \frac{\Pr (\tilde{Y}_B > u e_B, \tilde{Y}_{Bu'} \leq u e_{Bu'})}{\Pr (\max_{j=1,\ldots,m} \tilde{Y}_j > u)}
\]

\[
= \sum_{B \in \mathcal{P}, |B| = j} \lim_{u \to \infty} \frac{u \Pr (\tilde{U}_B > u e_B, \tilde{U}_{Bu'} \leq u e_{Bu'})}{u \Pr (\max_{j=1,\ldots,m} U_j > u)}.
\]

We therefore deduce by Lemma 4 that

\[
\lim_{\eta \to 0} \Pr (N_{mt}(\eta) = j | N_{mt}(\eta) \geq 1) = \sum_{B \in \mathcal{P}, |B| = j} \int_0^\infty \Pr (\tilde{U}_B > \gamma e_B, \tilde{U}_{Bu'} \leq \gamma e_{Bu'}) d\gamma \int_0^\infty \Pr (\max_{j=1,\ldots,m} U_j > \gamma) d\gamma = \sum_{B \in \mathcal{P}, |B| = j} p_B.
\]

### 6.2 Appendix 2

Let us first recall Breiman’s Lemma.

**Lemma 5** Suppose that \(X\) and \(Y\) are two independent random variables such that \(\Pr (X > x)\) is regularly varying of index \(-\alpha, \alpha \geq 0\), and that \(Y\) is nonnegative satisfying \(\mathbb{E}[Y^{\alpha+\varepsilon}] < \infty\) for some \(\varepsilon > 0\). Then

\[
\Pr (Y X > x) \sim \mathbb{E}[Y^\alpha] \Pr (X > x)
\]

as \(x\) tends to infinity.

Using Breiman’s Lemma, the assumption that \(\varepsilon_{jt}\) has a lower (negative) tail than the marginal distribution of the market portfolio, the assumption that \(\beta_{jt}\) is nonnegative and independent of \(r_{Mt}\) with \(\mathbb{E}[(\beta_{jt}^-)^\alpha] < \infty\), we deduce that

\[
\Pr (r_{jt} < -r) = \Pr (-r_{jt} > r) = \Pr (-\beta_{jt}^- r_{Mt | r_{Mt} \leq 0} > r + \alpha \varepsilon + \beta_{jt}^+ r_{Mt | r_{Mt} > 0} + \varepsilon_{jt})
\]

\[
\sim \Pr (\beta_{jt}^- (-r_{Mt}) > r) \sim \mathbb{E}[(\beta_{jt}^-)^\alpha] \Pr (-r_{Mt} > r) \sim \mathbb{E}[(\beta_{jt}^-)^\alpha] c_{\varepsilon_{Mt} - r}^{-\alpha}
\]

as \(r\) tends to infinity. Therefore we have that

\[
q_{jt}(r^{-1}) \sim -r^{1/\alpha} \mathbb{E}[(\beta_{jt}^-)^\alpha] c_{\varepsilon_{Mt} - r}^{-1/\alpha}, \quad r \to \infty,
\]

and

\[
\Pr (r_{jt} < q_{jt}(r^{-1}) z_j^{1/\alpha}) \sim r^{-1} z_j^{-1}, \quad r \to \infty.
\]
Using Breiman’s Lemma once again, we derive that
\[
\Pr \left( \cup_{j=1}^{m} \{ r_{jt} < q_{jt}(r^{-1}) z_j^{1/\kappa} \} \right) = \Pr \left( \min_{j=1,\ldots,m} \frac{r_{jt}}{-q_{jt}(r^{-1}) z_j^{1/\kappa}} < -1 \right) \\
\sim \Pr \left( \min_{j=1,\ldots,m} \frac{r_{jt}}{-q_{jt}(r^{-1}) z_j^{1/\kappa}} < -r^{1/\kappa} \right) \\
\sim \Pr \left( \max_{j=1,\ldots,m} \frac{\beta_{jt}}{\{ \mathbb{E}[(\beta_{jt}^{-\kappa})^z_{CM-}]^{1/\kappa} z_j^{1/\kappa} \}} < -r^{1/\kappa} \right) \\
\sim \Pr \left( \max_{j=1,\ldots,m} \frac{\beta_{jt}}{\{ \mathbb{E}[(\beta_{jt}^{-\kappa})^z_{CM-}]^{1/\kappa} z_j^{1/\kappa} \}} > r^{1/\kappa} \right) \\
\sim \mathbb{E} \left[ \max_{j=1,\ldots,m} z_j^{-1}(\beta_{jt}^{-\kappa})^{z} / \{ \mathbb{E}[(\beta_{jt}^{-\kappa})^z_{CM-}]^{1/\kappa} z_j^{1/\kappa} \} \right] r^{-1}
\]
as \ r \ tends \ to \ infinity.

Finally, note that
\[
1 / F_{jt}(q_{jt}(r^{-1}) z_j^{1/\kappa}) \sim r z_j, \quad r \to \infty,
\]
and hence that
\[
\Pr \left( \cup_{j=1}^{m} \{ r_{jt} < q_{jt}(r^{-1}) z_j^{1/\kappa} \} \right) = \Pr \left( \cup_{j=1}^{m} \{ Y_{jt} > 1 / F_{jt}(q_{jt}(r^{-1}) z_j^{1/\kappa}) \} \right) \sim \Pr \left( \cup_{j=1}^{m} \{ Y_{jt} > r z_j \} \right)
\]
to conclude.

### 6.3 Appendix 3

Recall that \((\pi_i)_{i=1,\ldots,l}\) is the trivial partition and \(C\) is an Archimedean copula with generator \(\psi\) such that
\[
C(u) = \psi \left( \psi^{-1}(u_1) + \ldots + \psi^{-1}(u_m) \right).
\]
Therefore \(\Pr (U \leq z) = C(F(z_1),\ldots,F(z_m))\) where \(F\) is the probability distribution function of \(U\) satisfying \(\mathbb{E}[U^+] = 1\).

Now note that
\[
V_B^* = \int_{0}^{\infty} \Pr (U_B > \gamma e_B, U_{B^c} \leq \gamma e_{B^c}) \, d\gamma \\
= \int_{0}^{\infty} \Pr (\cap_{i \in B} \{ U_i > \gamma \} \cap \cap_{j \in B^c} \{ U_j \leq \gamma \}) \, d\gamma \\
= \int_{0}^{\infty} \left[ 1 - \Pr (\cup_{i \in B} \{ U_i \leq \gamma \} \cap \cap_{j \in B^c} \{ U_j \leq \gamma \}) \right] \, d\gamma - \int_{0}^{\infty} \left[ 1 - \Pr (\cap_{i \in B^c} \{ U_i \leq \gamma \}) \right] \, d\gamma \\
= \int_{0}^{\infty} \left[ 1 - \Pr (\cup_{i \in B} \{ U_i \leq \gamma \} \cap \cup_{j \in B^c} \{ U_j \leq \gamma \}) \right] \, d\gamma - \int_{0}^{\infty} \left[ 1 - \Pr (\cap_{i \in B^c} \{ U_i \leq \gamma \}) \right] \, d\gamma
\]
and that
\[
V^* = \int_{0}^{\infty} \Pr \left( \max_{j=1,\ldots,m} U_j > \gamma \right) \, d\gamma = \int_{0}^{\infty} \left[ 1 - \psi (m \psi^{-1} (F(\gamma))) \right] \, d\gamma.
\]
Moreover we have
\[
\int_{0}^{\infty} \left[ 1 - \Pr (\cap_{i \in B^c} \{ U_i \leq \gamma \}) \right] \, d\gamma = \int_{0}^{\infty} \left[ 1 - \psi (|B^c| \psi^{-1} (F(\gamma))) \right] \, d\gamma
\]
and

\[
\int_0^\infty \left[ 1 - \Pr\left( \bigcup_{i \in B} \{ U_i \leq \gamma \} \cap \bigcap_{j \in B^c} \{ U_j \leq \gamma \} \right) \right] \, d\gamma \\
= \sum_{k=1}^{|B|} (-1)^{k-1} C_{|B|}^k \int_0^\infty \left[ 1 - \psi \left( \left| B^c \right| + k \right) \psi^{-1} \left( F(\gamma) \right) \right] \, d\gamma.
\]

It follows that

\[
V_B^* = - \int_0^\infty \left[ 1 - \psi \left( \left| B^c \right| \psi^{-1} \left( F(\gamma) \right) \right) \right] \, d\gamma + \sum_{k=1}^{|B|} (-1)^{k-1} C_{|B|}^k \int_0^\infty \left[ 1 - \psi \left( \left| B^c \right| + k \right) \psi^{-1} \left( F(\gamma) \right) \right] \, d\gamma
\]

\[
= \sum_{k=0}^{|B|} (-1)^{k-1} C_{|B|}^k \int_0^\infty \left[ 1 - \psi \left( \left| B^c \right| + k \right) \psi^{-1} \left( F(\gamma) \right) \right] \, d\gamma
\]

and finally that

\[
p_B = \frac{V_B^*}{V^*} = \frac{\sum_{k=0}^{|B|} (-1)^{k-1} C_{|B|}^k \int_0^\infty \left[ 1 - \psi \left( \left| B^c \right| + k \right) \psi^{-1} \left( F(\gamma) \right) \right] \, d\gamma}{\int_0^\infty \left[ 1 - \psi \left( m \psi^{-1} \left( F(\gamma) \right) \right) \right] \, d\gamma}.
\]

Since \( p_B \) only depends in this case of \(|B|\) and \(|B^c|\), we deduce that

\[
P_j = \sum_{B \in \mathcal{P}, |B| = j} p_B(\sigma) = C_m \sum_{k=0}^j (-1)^{k-1} C_j^k \int_0^\infty \left[ 1 - \left( m - j + k \right) \psi^{-1} \left( F(\gamma) \right) \right] \, d\gamma.
\]
## 6.4 Appendix 4

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Table 5: For Asset 1 to 50, orders of the fitted ARMA($p, q$)-GARCH($p', q'$) models with $p$-values of Ljung-Box tests for the series of the squares of the studentized residuals (10 and 20 lags)
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Table 6: For Asset 51 to 100, orders of the fitted ARMA($p,q$)-GARCH($p',q'$) models with $p$-values of Ljung-Box tests for the series of the squares of the studentized residuals (10 and 20 lags)