Rosario Monter

PhD Candidate

University of Lausanne and

University of Lyon 1
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Preface
Part I

On default risk models:

A literature review
Part II

Valuing Sovereign Default Risk:  

A pricing formula and its empirical results for the Mexican external debt$^2$

$^2$Published at Banque & Marchés, 78, sept-oct 2005, pp. 24-33.
Abstract

Under a stochastic discount factor framework, this study presents a pricing formula that determines the sovereign default risk premium. In particular, we compute the default risk premium for the Mexican external debt by two stochastic discount factors: the Export-Import Ratio process and the Short Rate-Inflation Ratio process. The empirical performance of the pricing formula for the Mexican economy data shows that the default probability obtained in our formula is consistent with the empirical default probability implied by Brady bonds.

In the last three decades, the loans made between developed and developing countries have become a relevant component in the world economy. At the beginning of the 70’s, the total debt of developing countries was less than 100 billion dollars, but at the end of the 80’s it amounted to 1,180 billions of dollars. Every year the world debt amount increases and more countries become unable to meet their promises. For example, in August 1982, Mexico was the first debtor country to announce that it was unable to service its external debt of approximately 80 billion dollars. As a consequence, new rescheduling agreements and alternative financial instruments were born. Brady bonds were created in 1989 in order to restructure outstanding sovereign debt into liquid debt instruments traded in the market.

From this perspective a default risk analysis is crucial. Banks active in international trade are highly concerned in measuring such risk because of the investment decisions they have to face through their corporate projects or international portfolios. From the debtor’s point of view, understanding the process linked to default could help them to restructure their own debts by the issuing of alternative financial instruments traded on the market.

Using the Mexican economy as a context, the present study will show a pricing formula that determines the default risk premium associated to a loan issued to an emerging
country.

This paper is organized as followed: chapter one reviews some typical methods and models used to measure default risk in emerging countries; chapter two presents a pricing formula that determines the default risk premium associated to a loan issued to an emerging country; chapter three resumes main macro-economic factors to be considered; chapter four applies the proposed pricing formula to the Mexican economic data and finally chapter five concludes and addresses further questions.
Chapter 1

Risk to default: adding up the models

Default risk is defined as the failure to pay interest or principal debt promptly when due. If the loan is issued to an entity established in a country different from the lender’s residence, additional risks are incurred denominated country risk. It includes the risk of expropriation of dividends by local government, the risk that as result of a war or political events a firm may not be paid for its exports, the risk faced as a consequence of different legislations, geographical situations, economic conditions, etc. Sovereign risk refers to the risk that a government might default on its own external obligations.

There are several reasons a government cannot meet its obligations:

- Political risk: the government may be unwilling to pay because of internal political problems.

- Liquid risk: there is not enough foreign exchange money available when needed.
• *Transfer risk*: money cannot be transferred in or out of the country.

• *Currency risk*: risk related to the value of foreign currency changes.

• *Economical risk*: related to local economic situations.

It is clear that emerging countries are more vulnerable to fall into default than developed countries mainly due to its economic and political instability policies, but how to measure its sovereign default risk?

1.1 Measuring sovereign default risk

For several reasons it is crucial to determine how sovereign default risk can be measured. Banks active in international trade are highly concerned in measuring such risk because the investment decisions they have to face through corporate projects or in creating new portfolios. From the owing government’s point of view, understanding the process linked to default could help it to restructure its debt by the issue of alternative financial instruments traded on the market.

Different approaches have been used to measure sovereign default risk. They can be classified as qualitative and quantitative analysis.

**The qualitative analysis**

It consists of the interpretation of economic and political descriptive information that allows the analyst to make judgments about how risky the country is. It is mainly based on subjective and unmeasurable variables like social behavior, political rumors, civil unrest and media content. Unfortunately an objective default risk premium cannot be
deduced *only* from such analysis. For a revision of qualitative analysis see Cohen (1985), Eaton and Gersovitz (1981) or Ciarrapico (1992).

**The quantitative analysis**

It consists on making judgements inferred from real economic data by using statistical methods. Among them *Logit Analysis*, applied by McFadden (1973), Mayo and Barret (1977) and Morgan (1986), *Discriminant Analysis* by Frank and Cline (1971), Abassi and Taffler (1982) while *Principal Component Analysis* has been studied by Dhonte (1975). For a comparative analyses of all them see Saini and Bates (1978) or Collins (1982).

Although the statistical methods have the advantage of tractable risk measurement, there are drawbacks as well. It is not always possible to find enough reliable historical data or any data at all available, specially from emerging countries for that matter.

As a consequence of the common default of payments by debtor governments, credit institutions look for more effective methods to measure and price sovereign default risk. It is not enough to consider *only* econometric methods to deduce a default risk premium or *only* consider a qualitative analysis. Some mathematical credit risk models have been developed recently. Among them, two approaches are the most comun: the *Reduced Form* and the *Structural Approach* which are described below.

**1.2 Mathematical credit risk models**

**The Reduced Form**

Developed by Jarrow and Turnbull (1995), this approach models default as an unpredictable event governed by a hazard-rate process $h_t$ and a fractional expected loss $L_t$
if default occurs at some time $t$. Thus default risk is based on an intensity exogenous rate process $R_t$

$$R_t = r + h_t L_t$$

where $r$ denotes the risk free interest rate. If a contingent claim instrument pays a fixed amount $K$ at maturity date in case of default, the risk premium is calculated as the expected value of the payoff $K$ discounted by the intensity process $R_t$ under a risk neutral probability measure $Q$ given the information $I_t$. Therefore, the risk premium becomes

$$P_t = E^Q [\exp (-R_t dt) * K | I_t]$$

Most recent theoretical research is due to Jarrow, Lando and Turnbull (1997) and Duffie and Singleton (1999). The main message of this approach is that the rate process already contains the default process.

**The Structural Approach**

Under this approach, the default risk premium is viewed as a *put option* contract. Following Merton (1974), in corporate default risk, the firm issues debt and equity securities and invests them as a tradable asset, then a default event is produced if the asset value $S_t$ falls below a specified value $K$ at a terminal fixed date $T$. The default risk premium $P_t$ is given by the expected value, under a risk neutral probability measure $Q$, of the payoff $\max \{K - S_T, 0\} = [K - S_T]^+$ discounted by the risk-free interest rate process $r_s$. The default risk premium or *put option* price is

$$P_t = E^Q \left[ \exp \left( - \int_t^T r_s ds \right) [K - S_T]^+ \right]$$

Under the assumption that the asset’s value follows a stochastic diffusion process
like the following:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad \sigma \text{ constant}
\]

then the default risk premium becomes the Black and Scholes' put option pricing formula.

Among the literature that follows the structural approach Hull and White (1992), Hull and White (1995), Longstaff and Schwartz (1995) are the most representative.

Sovereign default risk can be modelled similarly by defining \( K \) as the fixed amount of debt borrowed by the emerging country to be paid at maturity date \( T \). Here \( S_t \) would represent the capacity or willingness of payment made by the debtor\(^1\). Therefore, the defaultable amount at maturity date is \( [K - S_T]^+ \), thus the default risk premium becomes equivalent to the price of an European put option contract.

Surprisingly, given the clear relationship between option contracts and default risk insurance, option pricing theory was used very late to evaluate sovereign default risk. Pioneer work was due to Claessens and Wijnbergen (1990), Chesney and Morisset (1992), and Claessens and Pennacchi (1996). This relationship suggests that sophisticated option pricing theory can be “adapted” in evaluating corporate or sovereign default risk. These similarities have already been investigated by some authors. For instance, Saa Requejo and Santa-Clara (1999) define corporate default as the first time the solvency of the company falls down some specific level \( K^* \), then the defaultable bond premium is equivalent to the price of an American put option contract with payoff \( [K^* - S_T]^+ \).

A recent methodology that has been applied to option pricing models is the use of stochastic discount factors. They were introduced in asset pricing models by imposing a

\(^1\)There is some literature that emphasize the difference between the debtor’s payment capacity and the willingness to pay. For our purpose no difference will be made.
factor structure to a collection of asset returns or payoffs to describe their joint distribution at a point time. This methodology have been extended to option pricing models\textsuperscript{2} and it can be adapted to price sovereign default risk.

It is clear that all improvements developed in option pricing theory can be adapted to price corporate and sovereign default risk. Of course, there are some differences between derivative instruments, corporate debt and sovereign debt, but a unified methodology of evaluating default risk can be applied by introducing appropriate parameters according to the debt’s nature and the economic environment where the debt is issued.

Under the *Structural Approach*, the contribution of this paper consists of introducing a suitable stochastic discount factor and debtor’s capacity of payment process that allow us to calculate the sovereign default risk premium. Moreover, the capacity of payment process could be characterized according to the different states of the economy by either a random Markov-state transition matrix, or a deterministic criteria, whatever results appropriate from macroeconomic data.

\textsuperscript{2}An excellent book that presents asset pricing theories under a unified stochastic discount factor approach is Cochrane (2001).
Chapter 2

Pricing sovereign default risk

Following Cochrane (2001), the most basic pricing equation of any asset is given by the expected value of its payoff in a future date, actualized by some stochastic discount factor $E_t [m_{t,T} \cdot \text{payoff}]$.

Since we don’t know in advance the debtor’s capacity of payment at some given date, we can assume that it follows a stochastic process $S_t$. Because we are interested in the default event only at terminal date and not before, the default risk premium $P_t$ can be viewed as an *European put option* contract. The random payoff is given by the positive difference of the fixed debt contracted $K$ minus the random debt payment $S_T$ at maturity date $T$.

The pricing equation can be written as

$$P_t = E_t [m_{t,T} \cdot (K - S_T)^+]$$

(2.1)

where $m_{t,T}$ is the stochastic discount factor\(^1\).

\(^{1}\)Harrison and Kreps (1979) showed that the existence of a positive stochastic discount factor $m_{t,T}$ is
2.1 Pricing with a stochastic discount factor

One can assume that the pricing probability does not depend on the initial payment capacity $S_t$, thus the pricing function is homogeneous of first degree with respect to $(S_t, K)$. Lets denote by $k = \frac{K}{S_t}$ the ratio between the contracted debt amount and the initial payment capacity.

Equation 2.1 becomes

$$P_t = S_t \mathbb{E}_t \left[ m_{t,T} \cdot \left( \frac{K - S_T}{S_t} \right)^+ \right]$$

$$= S_t \mathbb{E}_t \left[ m_{t,T} \cdot \left( \frac{K}{S_t} \right) \cdot 1_{K > S_T} - m_{t,T} \cdot \left( \frac{S_T}{S_t} \right) \cdot 1_{K > S_T} \right]$$

$$= S_t \left\{ k \mathbb{E}_t \left[ m_{t,T} \cdot 1_{k > \frac{S_T}{S_t}} \right] - \mathbb{E}_t \left[ m_{t,T} \cdot \left( \frac{S_T}{S_t} \right) \cdot 1_{k > \frac{S_T}{S_t}} \right] \right\}$$

$$= S_t \left\{ k \mathbb{E}_t \left[ m_{t,T} \cdot 1_{\log k - \log \frac{S_T}{S_t} \geq 0} \right] - \mathbb{E}_t \left[ m_{t,T} \cdot \frac{S_T}{S_t} \cdot 1_{\log k - \log \frac{S_T}{S_t} \geq 0} \right] \right\} \tag{2.2}$$

Extending to a multiperiod framework, the stochastic discount factor and the payment’s capacity can be rewritten as

$$m_{t,T} = \prod_{\tau=t}^{T-1} m_{\tau,\tau+1} = \exp \left( \sum_{\tau=t}^{T-1} \log (m_{\tau,\tau+1}) \right)$$

$$\frac{S_T}{S_t} = \prod_{\tau=t}^{T-1} \frac{S_{\tau+1}}{S_{\tau}} = \exp \left( \sum_{\tau=t}^{T-1} \log \left( \frac{S_{\tau+1}}{S_{\tau}} \right) \right)$$

A more interesting model arises when information on the economic state at a certain time $t$ is considered. Lets denote by $U^T = (U_{\tau})_{1 \leq \tau \leq T}$ the path of economic state variables. We assume that the conditional variables $(m_{\tau,\tau+1}, \frac{S_{\tau+1}}{S_{\tau}})_{1 \leq \tau \leq T-1}$ are serially independent given the path of states variables $U^T = (U_{\tau})_{1 \leq \tau \leq T}$. This path can be characterized in several ways, in particular, a Markov switching process can be adapted allowing randomness in the equivalent to the absence of arbitrage.
duration that a given state remains or change. This characterization results appropriate when one wish to introduce recession and expansion macro economic periods, for instance.

To simplify notation, let us define

\[ A \equiv \sum_{\tau=t}^{T-1} \log (m_{\tau,\tau+1}) , \quad B \equiv \log k - \sum_{\tau=t}^{T-1} \log \left( \frac{S_{\tau+1}}{S_{\tau}} \right) \]

then

\[ m_{t,T} = \exp(A) , \quad \text{and} \quad \frac{S_T}{S_t} = k \exp(-B) . \]

Thus, equation 2.2 becomes

\[ P_t = kS_t \{ E_t [\exp(A) \cdot 1_{B \geq 0}] - E_t [\exp(A) \cdot \exp(-B) \cdot 1_{B \geq 0}] \} \]

By the independent assumption and applying the law of iterated expectations we now have

\[
\begin{align*}
P_t &= kS_t \cdot \{ E_t \{ E_t [\exp(A) \cdot 1_{B \geq 0} | U^T] \} - E_t \{ E_t [\exp(A - B) \cdot 1_{B \geq 0} | U^T] \} \} \\
&= K \cdot E_t \{ E_t [\exp(A) \cdot 1_{B \geq 0} | U^T] - E_t [\exp(A - B) \cdot 1_{B \geq 0} | U^T] \}
\end{align*}
\]

which proves the following result:

**Proposition 1** Assuming that the variables \((m_{\tau,\tau+1}, \frac{S_{\tau+1}}{S_{\tau}})_{1 \leq \tau \leq T-1}\) are conditionally serially independent given the path of state variables \(U^T = (U_\tau)_{1 \leq \tau \leq T}\), the default risk premium becomes

\[ P_t = K \cdot E_t \{ H(A, B, U) - G(A, B, U) \} \quad \text{(2.3)} \]

where

\[ H(A, B, U) = E_t [\exp(A) \cdot 1_{B \geq 0} | U^T] \]
\[ G(A, B, U) = \mathbb{E}_t \left[ \exp (A - B) \cdot 1_{B \geq 0} \mid U^T \right] \]

\[ A = \sum_{\tau = t}^{T-1} \log (m_{\tau, \tau+1}) \]

\[ B = \log k - \sum_{\tau = t}^{T-1} \log \left( \frac{S_{\tau+1}}{S_{\tau}} \right), \quad k = \frac{K}{S_t} \]

In order to obtain a pricing formula, the conditional joint probability distribution of the stochastic discount factor and the debtor’s payment capacity process \( \left( m_{\tau, \tau+1}, \frac{S_{\tau+1}}{S_{\tau}} \right) \) must be specified.

### 2.2 The pricing formula

In order to apply the previous pricing framework in a sovereign default risk context, we have to consider macroeconomic variables that could explain the debtor’s capacity of payment. Let’s denote by \( m_t \) the stochastic discount factor that will serve to price default risk and by \( S_t \) the underlying debt service process. If we assume that the local economy can be characterized by two random states: a solvent and a close-to-default state\(^2\), then debtor’s capacity of payment \( S_t \) can be written as a two-state Markov switching process:

\[ \log \frac{S_{t+1}}{S_t} = \mu_1 U_{t+1} + \mu_2 (1 - U_{t+1}) + [\sigma_1 U_{t+1} + \sigma_2 (1 - U_{t+1})] \varepsilon_{s,t+1} \]

where the random variable \( U_{t+1} \) represents the economic state at time \( t + 1 \). This variable takes value one with probability \( \pi_1 \) when the economy is in the solvent state, or value two with probability \( \pi_2 \) when it is in the close-to-default state. The probability that the economic state variable switch from state \( i \) to state \( j \) is given by \( p_{i,j} = \Pr[U_{t+1} = j \mid U_t = i] \), where \( p_{i,j} = 1 - p_{i,i} \) for \( i \neq j \), and \( \pi_i = \frac{1 - p_{i,j}}{2 - p_{i,i} - p_{i,j}}, \quad i, j = 1, 2 \).

\(^2\)The reason that we do not consider a default state in the model is that usually when a default occurs, the default amount is renegotiated or restructured.
We assume that the variables \( \left( \log m_{t+1}, \log \frac{S_{t+1}}{S_t} \right) \) follow two correlated stochastic processes:

\[
\begin{align*}
\log \frac{m_{t+1}}{m_t} &= \mu_m (U_t) + \sigma_m (U_t) \varepsilon_{1,t} \\
\log \frac{S_{t+1}}{S_t} &= \mu_S (U_t) + \sigma_S (U_t) \varepsilon_{2,t}
\end{align*}
\]

where \( (\varepsilon_{1,t}, \varepsilon_{2,t}) \) follows a bivariate standard normal distribution with correlation coefficient \( \rho_{m,S} \).

**Proposition 2** Under conditions of proposition one and assuming that the conditional probability distribution of the variables \( \left( \log m_{t+1}, \log \frac{S_{t+1}}{S_t} \right) \) given the economy state \( U^\tau \) at time \( \tau = 1, \ldots, T \) is a bivariate normal distribution with parameters

\[
E \left( \begin{array}{c} \log m_{t+1} \\ \log \frac{S_{t+1}}{S_t} \end{array} \right) = \begin{bmatrix} \mu_{m_{t+1}} \\ \mu_{S_{t+1}} \end{bmatrix}
\]

and

\[
\text{Var} \left( \begin{array}{c} \log m_{t+1} \\ \log \frac{S_{t+1}}{S_t} \end{array} \right) = \begin{bmatrix} \sigma^2_{m_{t+1}} & \sigma_{m_{t+1}S_{t+1}} \\ \sigma_{m_{t+1}S_{t+1}} & \sigma^2_{S_{t+1}} \end{bmatrix},
\]

the default risk premium \( P_t \) becomes

\[
P_t = K \cdot E_t \left[ \exp \left( \mu_A + \frac{1}{2} \sigma^2_A \right) \Phi \left( d_1 \right) - \exp \left( \mu_A - \mu_B + \frac{1}{2} \left( \sigma^2_A + \sigma^2_B - 2 \sigma_{AB} \right) \right) \Phi \left( d_2 \right) \right]
\]

(2.4)

where

\[
\begin{align*}
d_1 &= \frac{\mu_B + \sigma_{AB}}{\sigma_B}, \quad d_2 = d_1 - \sigma_B \\
\mu_A &= \sum_{\tau=t}^{T-1} \mu_{m_{\tau},r+1} \\
\mu_B &= \log \left( \frac{K}{S_t} \right) - \sum_{\tau=t}^{T-1} \mu_{S_{\tau},r+1} \\
\sigma^2_A &= \sum_{\tau=t}^{T-1} \sigma^2_{m_{\tau},r+1}
\end{align*}
\]
\[ \sigma_B^2 = \sum_{\tau=t}^{T-1} \sigma^2_{S_{\tau},r+1} \]

\[ \sigma_{AB} = \rho_{AB} \left( \sum_{\tau=t}^{T-1} \sigma^2_{m_{\tau},r+1} \right)^{1/2} \left( \sum_{\tau=t}^{T-1} \sigma^2_{S_{\tau},r+1} \right)^{1/2} \]

See Appendix for the proof.

In order to apply this pricing framework, the processes \( m_{t+1} \) and \( S_{t+1}/S_t \) must be specified according to the economic context. In particular, we consider macroeconomic variables that could explain the capacity of payment for the Mexican economy. Before doing that, a short description of modern Mexican economy is presented in the following section.
Chapter 3

The Mexican economy default risk

3.1 Watching the money fly

In the 70’s, the Mexican economic situation was favorable. High export growth rates especially in petroleum, minerals, and other natural resources were the typical warranty for loans. Unfortunately, that period was also characterized by a fiscal deficit accumulation brought on by the government’s incredible spending program based on the notion that oil production would be the only source of income to pave the way to “the Mexican miracle.”

A crisis became inevitable when interest rates rose and oil prices fell in 1982 leaving Mexico devastated by this international economic situation. In the same year the elections gave way to the De la Madrid government which prioritized the elimination of the trade deficit by cutting social expenditure and public investments. To achieve these goals the Peso was devalued in 1983, which led to high importation costs and inflation jumped to 125%. The purchasing power was considerably reduced and as a consequence in 1984 Mexico was
hit with a recession.

In 1985 with the announcement that Mexico would joint the *General Agreement on Trade Tariffs (GATT)* created a favorable economic atmosphere. In 1986 Mexico’s government was oriented towards a greater integration in the world economy by opening its doors much more than ever before. In exchange, Mexico was able to reschedule its debt and new money was borrowed from the *World Bank* and the *International Monetary Fund (IMF)*. However fiscal deficits and low petroleum prices led to even higher inflation levels in 1987 and early 1988, when it reached its highest historic level at almost 200%.

In December 1987 the Mexican government and representatives of major companies negotiated the *Pacto*, a fiscal policy program that planned to freeze the minimum wage, the cost of public services and the exchange rate. As a consequence, at the end of 1988 inflation was reduced to 52%. However, the freeze of the nominal exchange rate and thus the artificially low inflation levels led even bigger gap between the real exchange rate and Mexico’s frozen rate. The free flow of imports, the artificial exchange rate and the electoral political crisis led to a capital flight of 2.5 billion dollars.

In 1988, the new government negotiated another fiscal program, the *PECE*, where tax rates were lowered as well cuts made once again to social expenditure. The fixed exchange rate was replaced by a gradual devaluation process.

In 1989 it was announced that Mexico’s trade balance was not sufficient enough to service its debt, therefore new negotiations with the international financial community began. In March 1989, the *Brady plan* was announced: debt and interest were transformed into liquid debt instruments called *Brady bonds* backed by the U.S. government. As a
consequence debt payments where reduced when in 1990 the *Brady plan* was put into effect.

The beginning of the 90’s was characterized by neo-liberalism economic standards. In 1993 the *North America Free Trade Agreement (NAFTA)*, was signed between Canada, Mexico and the United States. The Mexican government thought that this agreement would lead to economic growth due to an influx of capital from the North. It offered excellent conditions to North American companies to make investments in Mexico, allowing them to profit from low salaries, no trade barriers, and little or no environmental regulations.

In conformity with the *IMF*’s social restructuring program, during 1992 and 1993 around one thousand state-owned enterprises were privatized and the government’s budget for social service programs such as health and education was drastically reduced.

Not all political and social groups saw the same advantages in neo-liberalism: the investment of foreign companies had not meant any increase in the standard of living nor in the purchasing power of workers. Privatization was a dirty process where politician’s friends and relatives benefited from the private auction of the country’s assets. The gap between rich and poor was now more than ever critical: while twenty-seven businessmen controlled 30% of the GDP, the working class survived on one dollar a day.

These economic and political factors led to the insurgence of important social movements. In 1994 the *Zapatista’s National Liberation Army (EZLN)* composed mainly of indigenous people from the south of Mexico started a rebellion in Chiapas and quickly spread to other states. It was the beginning of a political crisis. The long-awaited investments did not appear either and high interest rates attracted only speculative capital investments.

The 1994 presidential elections were precluded by a series of violent acts including
the assassination of a presidential candidate. For fear of losing these elections the current Mexican head of State, Gortari, refused to devalue the Peso. The new president elect was obligated to float the currency from 3.5 to 7 pesos per dollar. This first devaluation was followed by many more. The economic and political situation lead to a flight of capital based on fear and speculation resulting in 2.5 billion dollars by the end of the year.

In December 1994 the stock market dropped 24%, hundreds of companies closed down and more than 250,000 Mexicans lost their job. The repercussions from Mexican crash throughout Latin American markets were called the Tequila Effect. The December crash led to three years of economic depression, the worst to hit Mexico since 1920. This situation led to high growth of the informal economy.

In 1995 Mexico received a loan from the IMF and from the U.S. government to the value of 50 billion dollars. The loan went to pay off private investors who had speculated in Mexico. The Mexican government hoped to convince investors that the crisis in Mexico was over and their capital should stay.

During 1996 and 1997 government privatization continued: ports, railroads, airports, telecommunications, natural gas distribution, electricity and some petrochemical sectors were open to private investors. The Mexican President signed free trade agreements with Bolivia, Chile, Costa Rica, Nicaragua and Venezuela and initiated negotiations with Belize, Ecuador, El Salvador, Guatemala, Honduras, Panama, Peru, and the Mercosur market of Argentina, Brazil, Paraguay and Uruguay. In December 1997 Mexico signed the Agreement for Economic Association, Political Dialogue and Cooperation with the European Union.
In 1998, despite privatization and the neo-liberal economy, the high dependence of the federal budget on oil revenues (accounting for 40 %), and the drop of international oil prices, led to a tax increase and a drastic reduction of social expenditures. Again the education and health sectors were the most affected. Government imposed price increases on gasoline and diesel fuel, which is used to move industrial freight in Mexico. Also, it eliminated the official subsidy to tortilla manufacturers\(^1\) with the resulting increase of 20% in its price.

In 1998 the Savings Protection Bank Fund (Fobaproa) absorbed the bad portfolios of the national banks which granted very large loans without sufficient collateral\(^2\).

In 1999, after long sessions in the Congress, the executive’s initial proposal of converting Fobaproa’s liabilities into a public debt was approved. The official logic was that if the bad debt portfolio now paid by Fobaproa was not formalized into public debt, the banking system risked another capital flight and the Mexican economy could face another financial collapse. Fobaproa’s liabilities amounted up 61 billion pesos, equivalent to 15% of the gross domestic product. The financial crisis of 1999 was inevitable and the domestic private sector now has no access to credit which led several local companies to close.

The social discontent increased and in the year 2000, fraudulent elections were no longer possible. It was the first time after 71 years of the Institutional Revolutionary Party (PRI) government, that a candidate of the opposition party was recognized to be officially elected.

The structural reforms proposed by the new government for its mandate period of

\(^{1}\)Tortilla is the main food staple of the Mexicans.

\(^{2}\)Some bank directors are persecuted for white collar crime.
2000-2006 are: reduce social expenditures, increase substantially the tax revenue\(^3\), open the
electricity, natural gas, telecommunication and petrochemical industries completely to the
private sector, guarantee transparency of the Mexican financial system, enforce international
agreements such as the *Plan Puebla-Panama*, initiative which involves the eight southern
states of Mexico and seven nations of Central America, and to promote *NAFTA* expansion
by allowing free flow of labour. This “terrific” plan may have led to *Moody’s Investors
Service* to upgraded Mexico’s credit rating to *Investment Quality* in July 2000.

In February 2001 the *World Bank* approved a loan of 1.5 billion dollars to be spent
to “guarantee macroeconomic stability” by implementing strict fiscal reforms and opening
to private sector the few government controlled industries left like *Petroleum of Mexico
(PEMEX)*.

### 3.2 Sovereign default information

What data should be used to model sovereign default? Among practitioners of
rating agencies, relevant data can be classified into the following categories:

*The Debt Position*

It consists on the country’s current position and past performance on foreign obligations. Figure 3.1 graphs the external public debt issued to Mexico as well as the consolidated debt amount from March 1981 to March 2001. Figure 3.2 shows them as a percentage of the GNP.

*The Local and External Economy Performance*

\(^3\)Including tax in food and medicaments.
This information evaluates how the country is performing both domestically and in international markets. The total export and import amount including oil for the period of March 1981 to June 2001 are illustrated in Figure 3.3. The export-import ratio is illustrated in Figure 3.4. The GNP performance for the same period is depicted in Figure 3.5. The U.S dollar exchange rate is in Figure 3.6. The inflation rate and government short interest rate CETES\(^4\) from February 1985 to June 2001 are depicted in Figure 3.7 while the CETES-inflation ratio is in Figure 3.8.

*Liquidity*

The country should have access to a sufficient amount of liquidity to avoid default

\(^4\)Risk-free government guaranteed deposit certificates for a period of 28 days.
on foreign debt. Unfortunately there is not enough public historical data available of the Mexican international reserves. Figure 3.9 graphs the international reserves of Mexico from January 2000 to May 2001.

**Political stability**

It is important to assess the country’s political stability because the political regime may dramatically change the country’s priorities and policies and thus affect its willingness to service obligations contracted under the old regime. We need information on what policy it is pursuing and how these policies are implemented. Among this information it is important to know that CETES rates were liberated at the beginning of 1990 allowing the market mechanism rather than the government to determine its value. It is also important
to note that the U.S. dollar exchange rate was controlled by the Mexican government until 1994.

There are three official sources of Mexican public economy data:

Figure 3.7: CETES and Inflation

Figure 3.8: CETES / Inflation Ratio

Figure 3.9: International Reserves (million of dlls)
Chapter 4

Empirical results

The debtor’s capacity process, $S_t$ is given by the amount of debt that has been consolidated at time $t$. This process is illustrated in Figure 3.1.

Basically there are three financial sources for Mexican government to paid its external debt:

- taxes
- the trade surplus (mainly due to oil exportations)
- new debt: like issuing CETES or by new international borrowing.

Reviewing the modern economy history of Mexico (1980 - 2000), there is no evidence that the Mexican government desires to develop its own Industry. Loans have been used to guarantee macroeconomic variables to attract foreign investors hoping that they arrive to Mexico to bring the economic progress instead to activate local economy. Local banks do not achieve the functions for which have been created: there is no credit at all available for new companies and what is even worst, already existing companies are disap-
pearing. This lead most of the population to remain in the informal economy and therefore
taxes do not account significantly to meet external obligations.

Among the information available, there are two candidates to perform as a discount factor \( m_t \):

- The *Short Rate-Inflation ratio process*, denoted by \( m^r_t \) which consist on the monthly
government interest rate \( CETES \) divided by the monthly inflation rate.

- The *Export-Import Ratio process*, denoted by \( m^e_t \) which consist on total exports (in-\ncluding oil) divided by total imports.

The model presented in section three considers the path of economic state variables
at a certain time \( t \) denoted by \( U^T = (U_\tau)_{1 \leq \tau \leq T} \). Although this path can be characterized
by a Markov switching process, the available data of the Mexican government interest rates
suggests to characterize the Mexican economy in two states: a “fixed state” from February
1985 to December 1989 where CETES rates were fixed by the government, and a “liberated
state” from January 1990 to June 2001 when the market determined their value.

Note that the debt’s structure also changed: since 1990, when the Brady plan
was put into effect, until now, Brady bonds restructure external debt into tradable debt
instruments. Even more, Mexican economy started to be effectively open since 1990.

Because the currently state of the Mexican economy is a “liberated state” and the
probability of switch or go back to a “fixed state” is almost zero given the economic, political
and historical circumstances, the empirical test will be solely based on data from January
1990 until now.
Figures 4.1 and 4.2 plot the Short Rate-Inflation ratio process for the two periods. Figures 4.3 and 4.4 plot the Export-Import Ratio process for the same two periods.

The empirical experiment is performed as follows:

First, the parameters of the processes $(S_t, m_t^e)$ and $(S_t, m_t^i)$ are estimated using Monthly data from January 1990 to April 2001, the “liberated state”, presented in Table 4.1 with parameters $\hat{\alpha} = \frac{1}{nh} \sum_{k=1}^{n} r_k(h), \hat{\sigma}^2 = \frac{1}{nh} \sum_{k=1}^{n} (r_k(h) - \hat{\alpha}h)^2$, $\hat{\mu} = \hat{\alpha} + \frac{1}{2}\hat{\sigma}^2$, $r_k(h) \equiv \log\left(\frac{m_t}{m_{t-1}}\right)$ and $\hat{\sigma} = \left(\frac{1}{nh} \sum_{k=1}^{n} (r_k(h) - \hat{\alpha}h)^2\right)^{1/2}$.

Using these parameters, the default risk premium of equation 2.4 is calculated by
considering as the discount factor both the \textit{Export-Import Ratio} and the \textit{CETES-Inflation Ratio}. The initial debt service $S_0$ equals to the fixed debt issued $K$, i.e. $k = K/D_0 = 1$. The results are presented in Table 4.2. They may be interpreted as the default risk premium for each dollar borrowed. Since it is a \textit{pure} premium, in the sense that only the default risk is taken into account and not other costs, it can be interpreted as the proportion to default for each dollar assured, that is the probability of default.

Does this theoretical default probability reflect the market view? An interesting experiment could be based on estimate the empirical default probability inferred from some
Debtor’s Capacity

<table>
<thead>
<tr>
<th></th>
<th>Debtor’s Capacity $S_t$</th>
<th>Export/Import $m_t^e$</th>
<th>CETES/Inflation $m_t^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>-0.028882128</td>
<td>-0.003773099</td>
<td>-0.015952426</td>
</tr>
<tr>
<td>$\hat{\sigma}^2$</td>
<td>0.005803187</td>
<td>0.00430551</td>
<td>0.0443485550</td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>-0.031783722</td>
<td>-0.005925854</td>
<td>-0.038126703</td>
</tr>
<tr>
<td>$\hat{\sigma}$</td>
<td>0.07617865</td>
<td>0.065616386</td>
<td>0.2105909660</td>
</tr>
<tr>
<td>$\rho_{mS}$</td>
<td>correlations:</td>
<td>0.643266437</td>
<td>-0.417851367</td>
</tr>
</tbody>
</table>

Table 4.1: Parameter Estimation. Monthly data from January 1990 to April 2001

<table>
<thead>
<tr>
<th></th>
<th>Export/Import $m_t^e$</th>
<th>CETES/Inflation $m_t^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_t^e = 0.03994234$</td>
<td></td>
<td>$P_t^r = 0.03422552$</td>
</tr>
</tbody>
</table>

Table 4.2: Default Risk Premium

Tradable debt instruments and compare our “theoretical default risk” with the “market default risk”. In this direction, Izvorski (1998) estimates the default probability implicit in Brady bonds using data for the period of January 1994 to November 1996 for seven countries. He found a mean default probability for Mexico of 0.0307 which is consistent with the results presented in this study.

Another paper that estimate the Mexican payment capacity implied from market prices is due to Claessens and Pennacchi (1996). They construct a pricing model that takes into account the specific terms of the debt agreement and the presence of third-pard guarantee (oil price recapture). However, when applying their model to Mexican Brady bonds they found that the estimate repayment capacity performs differently from the bond prices.

It is important to notice that the repayment capacity model of this paper differs from the model proposed by Claessens and Pennacchi (1996). In this paper the main
task is to identify a suitable stochastic discount factor (country specific) that summarizes the sources that most contribute to repay foreign debt. Among this sources, financial instruments that perform as a third-part guarantee are considered. In particular for the Mexican economy performance, the *Export-Import Ratio process* is proposed as a discount factor. Note that this process contains implicitly oil revenues.
Chapter 5

Conclusion

Based in a stochastic discount factor methodology, this study presented a pricing formula that determines a theoretical sovereign default risk premium. Although a Markov switching process have been introduced in the model, qualitative analysis of the Mexican economy lead us to consider only one state: the \textit{liberated state} from January 1990 to April 2001.

Two discount factors were considered to capture the Mexican repayment capacity: the \textit{Export-Import Ratio process} and the \textit{Short Rate-Inflation Ratio process}. The \textit{Export-Import Ratio process} reflects how the country is performing in the domestic and international markets. It also reflects the capacity of payment since the most credible financial source for Mexican government to paid its external debt is the trade surplus (mainly due to oil exportations). The second process, the \textit{Short Rate-Inflation Ratio process} contains relevant information about the structure of domestic risk-free real interest rates.

The empirical performance of the pricing formula for the Mexican economy data
shows that the theoretical default probability obtained in our model is consistent with the empirical default probability implied by Brady bonds obtained by Izvorski (1998).

The model proposed here becomes particularly useful when there is no sufficient historical data available to extract an implied repayment capacity from financial debt instruments traded in the market, like Brady bonds.

Further research will focus on the application of this methodology to other emerging economies: the main task is to identify a suitable stochastic discount factor (country specific) that summarizes the sources that most contribute to repay foreign debt. Among this sources, financial instruments that perform as a third-part guarantee have to been incorporated.
Bibliography


Part III

Switching Parisians:

A default-risk model with macroeconomic effects
Abstract

When the value of a firm falls below a threshold level, debtholders have the right to force liquidation. However, bankruptcy laws may allow the firm to continue for a certain grace period. In order to value such liabilities, structural models assume that liquidation arrives when the distress period exceeds a given grace period, thus default risk is priced as a Parisian put option. Under this framework, this paper proposes a model where the parameters of the firm value process are driven by a continuous-time Markov chain process in order to account the macro economic regime conditions that may affect default risk. Thus, credit risk is modelled by incorporating both: a) a stochastic distress clock that accounts for a grace period before bankruptcy (Parisian option), and b) a Markov regime-switching model able to capture main macro-economic effects.

Extensive research work has been dedicated to modelling default and credit risk. Historically, two modelling approaches have been developed and used among practitioners and researchers: the reduced approach and the structural approach.

The reduced approach, introduced by Jarrow and Turnbull (1995), models default as an unpredictable event, governed by a hazard-rate process and a proportional expected loss if default occurs at some point in time. Thus, default risk is driven by an intensity exogenous rate process. If a contingent claim instrument pays a fixed amount at the maturity date in case of default, the risk premium is calculated as the expected value of such instruments, discounted by the intensity process under the risk neutral probability measure. Some of the theoretical research under this approach has been carried out by Jarrow, Lando and Turnbull (1997); Madan and Unal (1998) and Duffie and Singleton (1999). The main message of the reduced approach is that the stochastic rate process already contains the default process. While reduced models have been very convenient for estimation and computational issues, this approach does not aim to explain or to model the sources of default.
On the other hand, the *structural approach* introduced by Merton (1974) models default risk by taking into account endogenous structural variables that drive the payment capacity of the debtor. In this setting, applying the results of Black and Scholes (1973), default risk premium is priced as an European put option contract. Under the framework of this contract, the debtor, which may be a firm or a sovereign entity, issues tradable debt and equity securities. Thus default is produced if at maturity time, the debtor's payment capacity process falls below a fixed barrier or threshold, which represents the covenant to protect debt holders.

Since then, several improvements on structural credit risk models have been developed. For example, the first passage time model of Black and Cox (1976) allows default to arrive at any time, not just at maturity. Furthermore, Leland (1994) and Leland and Toft (1996) define strategic default by considering an endogenous threshold, which is obtained as a result of maximizing the value of the firm.

Under the structural approach, Fan and Sundaresan (2000), Moraux (2002) and François and Morellec (2004) make a distinction between default and liquidation. Since there are several laws that protect the debtor in distress by allowing it to continue doing business for a certain grace period, liquidation arrives only if the debtor is on distress for a period of time longer than the given grace period. Under this setting, pricing credit risk may be viewed as pricing a *put down-in Parisian option*. Parisian options, introduced by Chesney, Jeanblanc and Yor (1997), are barrier options that may be activated (in) or deactivated (out), when the underlying process reaches the barrier and remain below (down) or above (up) the barrier for a period longer than the specified grace period.
In addition to the characterization of the time to default and liquidation, academic work has been extended to consider different dynamics of the interest rate and the underlying process. For example, Longstaff and Schwartz (1995) consider a stochastic interest term structure, while Zhou (1997, 2001) analyzes default when the underlying follows a diffusion and jump process.

Although there has been considerable progress in modelling default, little research has incorporated endogenously the macro economic characteristics that may affect the behavior of the debtor on a long term basis. This is of great importance when assessing the default risk of investment assets of pension funds, life insurance products or any financial instrument considered in the long term.

A simple way to take into account the macro economic variables that drive the payment capacity process is by assuming that the parameters are governed by a regime switching process. Hamilton (1989) introduced regime switching processes to financial econometrics and economists have, since then, found several applications. For example, Garcia and Perron (1996) applied it to model real interest rates and Bansal and Zhou (2002) use it to model the term structure of interest rates. Hardy (2001) considered a regime switching process to model the returns of stocks in the long term, while Boyle and Draviam (2007) price Asian and look-back options under regime switching dynamics.

The goal of this paper is to develop a structural default risk model that incorporates both: a *Markov regime switching process* that affects the dynamics of the debtor payment capacity process and a *stochastic distress clock* that accounts for a grace period before liquidation. It is assumed that the underlying stochastic process follows a geometric
Brownian motion with drift and volatility parameters that switch according to a hidden Markov process. The intuition behind this framework is that the market and the economy may switch from an expansion period to a recession period, for instance, affecting the debtor’s payment capacity. Thus, in a recession period, the debtor may experience low returns and high volatility, whilst in an expansion period, it may experience high returns with low volatility.

This paper is structured as follows: Section 2 describes the continuous-time Markov chain that governs the regime switching process. Then it introduces an occupational time random variable that represents the amount of time the process stays on a particular state over a fixed time period. Thus, it provides a result that allows to derive the moment generating function of the joint probability density function for the occupational times. A result that will be used later in order to price the option.

Section 3 describes the payoff of the debt holder and the equity holder assuming that default may arrive only at maturity time. Then it describes the dynamics of the risk-free asset and the risky asset governed by the regime switching process in terms of occupational times.

Section 4 discusses the market incompleteness due to the extra uncertainty added by the regime switching process. It shows that conditioning on the switching states trajectory, makes it easy to find a risk-neutral probability measure that allows us to price a European call option under the regime switching process. Therefore, the unconditional price is derived by using the results of Section 2.

Section 5 reviews the model of Fan and Sundaresan (2000) where the assumption
that default may arrive only at maturity time is relaxed. Meaning it liquidation time as the first time the firm’s asset value stays below the debt level for a consecutive period longer than a given grace period. Furthermore, it shows the connection with Parisian options created by Chesney et al. (1997).

Section 6 presents the Switching Parisians model that combines the regime switching process of Section 4 with the Parisian distress clock presented in Section 5. It also presents the mathematical formulation of such model and we discuss the economic interpretations.

Even though a closed-form formula may be derived by inverting the moment generating function of the joint density of the occupational times, and even it exits a closed-form solution for pricing Parisians options, it may be a difficult task to invert such functions. Thus, Section 7 provides a numerical implementation of the Switching Parisians model. Finally, Section 8 concludes.
Chapter 6

The regime switching process

6.1 Continuous-time Markov chain

Let \{C(t), t \geq 0\} be a homogeneous continuous-time Markov chain defined on a probability space \((\Omega, \mathcal{F}_t^C, P)\), with a discrete state space of \(N\) elements represented by \(E = \{e_1, ..., e_N\}\) where \(e_i = [0, ..., 0, 1, 0, ..., 0]'\), \(i = 1, ..., N \in \mathbb{N}\) is the indicator vector having one at the \(i\)-th element and zeros elsewhere, and the prime ' denoting transpose. Let \(\mathcal{F}_t^C = \sigma (\{C(t)\})\) denote the complete filtration generated by the process \(\{C(t), t \geq 0\}\) and \(p_{ij}(t)\) denotes the probability that a process presently in state \(i\) will be in state \(j\) at a time \(t\) later, thus

\[p_{ij}(t) \equiv \Pr (C(t) = e_j | C(0) = e_i); t \geq 0\]

The Markov property implies that

\[\Pr (C(t+s) = e_j | \mathcal{F}_s) = \Pr (C(t+s) = e_j | C(s)); \forall i, j \in N; 0 \leq s \leq t\]
while the homogeneous property implies

\[ p_{ij}(t) = \Pr(C(s + t) = e_j \mid C(s) = e_i); \forall i, j \in N; 0 \leq s \leq t. \]

Denote by \( p_i(t) = \Pr(C(t) = e_i) \), the probability to be in state \( i \) at time \( t \). Thus \( p(0) = (p_1(0), ..., p_i(0), ..., p_N(0))' \) denotes the vector of initial probabilities.

Define the transition probability matrix by \( P(t) = [p_{ij}(t)] \), a \( N \times N \) matrix.

Note that \( \{P(t), t \geq 0\} \) is a stochastic semi-group, which means:

a) \( P(0) = [p_{ij}(0)] = I \), where \( I \) is the identity matrix;

b) \( P(t) \) is a matrix with non-negative entries and rows sum one;

c)

\[
\begin{align*}
p_{ij}(t + s) & = \Pr(C(t + s) = e_j \mid C(0) = e_i) \\
& = \sum_k \Pr(C(t + s) = e_j \mid C(s) = e_k) \Pr(C(s) = e_k \mid C(0) = e_i) \\
& = \sum_k p_{ik}(s) p_{kj}(t)
\end{align*}
\]

which implies the Chapman-Kolmogorov equation

\[ P(s + t) = P(s) P(t); s, t \geq 0. \]

Define

\[ A = \lim_{\delta t \to 0} \frac{P(\delta t) - I}{\delta t} \]

where \( I \) is the identity \( N \times N \) matrix.
Then,
\[
\frac{dP(t)}{dt} = \lim_{\delta t \to 0} \frac{P(t + \delta t) - P(t)}{\delta t} = \lim_{\delta t \to 0} \left( \frac{P(\delta t)}{\delta t} - I \right) P(t) = AP(t)
\]

which is the forward Kolmogorov equation with solution given by
\[
P(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.
\]

The transition probability matrix \(A = [a_{ij}]\) is a \(N \times N\) matrix called the generator of the process. It satisfies:

a) the off-diagonal elements \(a_{ij}; i \neq j\) are finite and non negative, representing the rate of switching from state \(i\) to state \(j\). (Note that the time spent in state \(i\) before switching to state \(j\) is exponentially distributed with parameter \(a_{ij}\)).

b) the diagonal elements \(a_{ii} = -\sum_{i \neq j} a_{ij}\), in other words, rows sum zero.

Remark 3 A property of a continuous-time Markov chain \(\{C(t), t \geq 0\}\) is the following semi-martingale representation (See Elliott et al. 1995).
\[
C(t) = C(0) + \int_{0}^{t} AC(s) ds + M(t)
\]
where \(\{M(t), t \geq 0\}\) is a martingale process with respect to the filtration \(\{\mathcal{F}_t^C\}\).

6.2 The time spent process

A convenient way to describe a regime switching process is by defining the amount of time the continuous-time Markov chain spends on state \(i\) over the time interval \([0, t]\). Let
\( T_i(t) \) denote this random variable, which may be expressed as an indicator function or inner product of vectors by

\[
T_i(t) \equiv \int_0^t \mathbf{1}_{[C(s)=i]} ds \quad \text{or} \quad T_i(t) = \int_0^t \langle e_i, C(s) \rangle \, ds
\]

where \( \mathbf{1}_{[C(t)=i]} \) denotes the indicator function having the value of one if the chain is in state \( i \) at time \( t \), and zero otherwise. Obviously \( T_1(t) + T_2(t) + \ldots + T_N(t) = t \).

Define the process \( \tilde{T}_d(t), t \geq 0 \) to be the weighted sum of the time spent on the states \( i = 1, \ldots, n; \) for \( n \leq N \), during the time interval \([0, t]\). Equivalently,

\[
\tilde{T}_d(t) = \sum_{i=1}^n d_i T_i(t) \quad (6.1)
\]

where \( d_i \) are some given positive weights.

The following theorem characterizes the behavior of the regime switching by providing the moment generating function of \( \tilde{T}_d(t) \). (See Darroch and Morris (1968)).

**Theorem 4** The moment generating function of \( \tilde{T}_d(t) \) is given by

\[
G_{T_d(t)}(\theta) \equiv E \left( e^{\theta \tilde{T}_d(t)} \right) = p(0)' e^{t(\lambda + \theta D_1)} 1
\]

where \( p(0) \) is the vector of initial probabilities over the states \( i = 1, \ldots, N; \) \( 1 \) is a \( N \times 1 \) vector of ones and \( D_1 \) is a \( N \times N \) matrix defined as \( D_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \) with \( D \) being a \( n \times n \) diagonal matrix, \( n \leq N \); with elements in the diagonal being the weights \( d_i \) for \( i = 1, \ldots, n \).

**Remark 5** In particular, the moment generating function of the occupational time for a specific state \( i, T_i(t) \), may be obtained by replacing \( D_1 \) by an \( N \times N \) matrix with one on the \((i, i)\)-element and zeros elsewhere.
**Remark 6** To obtain the joint density function of \((T_1(t), ..., T_N(t))\), denoted by \(\phi(t_1(t), ..., t_N(t))\) it is sufficient to inverse the moment generating function \(G_{\tilde{T}_d(t)}(\theta)\), which may be a difficult task.

For example, the distribution function of the occupation time for a two-state continuous time Markov chain is provided below (see Pedler (1971)).

**Example 7** The distribution function of the occupation time \(T_1(t) = \tau\) for a two-state continuous-time Markov chain with generator \(A = \begin{bmatrix} -\lambda & \lambda \\ \nu & -\nu \end{bmatrix}\) is given by

\[
f(\tau,t) = e^{-\lambda \tau - \nu (t-\tau)} \{p_1 \delta(t-\tau) + p_2 \delta(\tau) \\
+ \left[ p_1 \left( \frac{\lambda \nu \tau}{t-\tau} \right)^{1/2} + p_2 \left( \frac{\lambda \nu (t-\tau)}{t-\tau} \right)^{1/2} \right] I_1 \left[ 2 \left( \frac{\lambda \nu \tau (t-\tau)}{t-\tau} \right)^{1/2} \right]
+ (p_1 \lambda + p_2 \nu) I_0 \left[ 2 \left( \frac{\lambda \nu \tau (t-\tau)}{t-\tau} \right)^{1/2} \right] \}
\]

where \(I_r(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+r}}{k!(k+r)!}\) is the modified Bessel function of order \(r\) and \(\delta(\ )\) is the Dirac’s delta function. Note that for this example \(p(0)' = [p_1(0), p_2(0)]\), \(D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\) and \(1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).
Chapter 7

Dynamics of the regime switching model

7.1 The riskless asset

Suppose that the market contains a riskless asset $B^0(t)$ whose risk-free interest rate evolves according to the continuous-time Markov chain $\{C(t); t \geq 0\}$. Let $r(C(t)) = (r, C(t)); r = (r_1, ..., r_N)^T; r_i > 0$ be the vector of instantaneous risk-free rates associated to each possible state $i = 1, ..., N$.

The dynamics of the riskless asset is described by

$$dB^0(t) = r(C(t)) B^0(t) \, dt$$

or equivalently,

$$B^0(t) = B^0(0) e^{\int_0^t r(C(s)) \, ds}$$
The integral may be written as
\[
\int_0^t r(C(s)) \, ds = \int_0^t \left( \sum_{i=1}^N r_i 1_{[C(s)=e_i]} \right) \, ds = \sum_{i=1}^N r_i \int_0^t 1_{[C(s)=e_i]} \, ds = \sum_{i=1}^N r_i T_i(t) = \bar{T}_r(t)
\]

Thus,
\[
B^0(t) = B^0(0) e^{\bar{T}_r(t)}
\]

where \( \bar{T}_r(t) \) is the weighted sum of the times spent on the states \( i = 1, \ldots, N \) over the interval \( [0, t] \) with weights equal to the instantaneous risk-free rate \( r_i \).

Applying theorem 1, the characterization of the riskless asset in terms of occupational times is derived:

**Proposition 8** The moment generating function of the log return of the bond prices is given by
\[
G_{\ln B^0(t)}(\theta) \equiv E \left( e^{\theta \ln(B^0(t))} \right) = E \left( B^0(t)^\theta \right) = E \left( e^{\theta \bar{T}_r(t)} \right) = G_{\bar{T}_r(t)}(\theta) = p(0)' e^{t(A+\theta D_r)} 1
\]

with \( D_r \) is the \( N \times N \) matrix with elements in the diagonal \( (r_i) \); \( r_i > 0 \) being the return rates on the riskless asset for \( i = 1, \ldots, N \).
7.2 The risky asset

7.2.1 Payoff

Under the classical credit risk model of Merton (1974), pricing credit risk is equivalent to price a European put option.

Assume that the firm is financed by equity and debt with face value $K$ at maturity time $T$ and assume that the firm can neither issue new debt nor equity. If at maturity time the firm value exceeds or equals the debt value $K$, then debt holders receive full payment and equity holders receive the remaining value $S(T) - K$. However, if the firm value falls below $K$, the firm will not be able to meet its obligations and debt holders, which have priority over equity holders, will receive the firm value $S(T)$, whilst equity holders receive nothing.

The payoff is summarized on the table below.

<table>
<thead>
<tr>
<th></th>
<th>No Default</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset value</td>
<td>$S(T) \geq K$</td>
<td>$S(T) &lt; K$</td>
</tr>
<tr>
<td>Debt holder</td>
<td>$K$</td>
<td>$S(T)$</td>
</tr>
<tr>
<td>Equity holder</td>
<td>$S(T) - K$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Therefore, under the structural classical approach, the debt holder may be covered for default risk by buying a European put position with payoff $[K - S(T)]^+$, where the underlying process is the asset’s value of the firm $S(t)$.

7.2.2 Dynamics

In order to accommodate the macro economic fluctuations that may affect the dynamics of the firm’s assets’ value, assume that the return rate and the volatility of the risky asset depend on a continuous-time Markov chain $\{C(t); t \geq 0\}$. 
Let \((\Omega, \mathcal{F}_t, P)\), be a complete probability space, where \(P\) is the real world or physical probability measure. Assume that the dynamics of the risky process \(\{S(t); t \geq 0\}\) is described by

\[
dS(t) = \mu(C(t)) S(t) \, dt + \sigma(C(t)) S(t) \, dW^P(t)
\] (7.2)

We assume that the continuous-time Markov chain \(\{C(t); t \geq 0\}\), defined on the filtration \(\mathcal{F}_t^C\) is independent of \(W^P(t)\), a standard Brownian motion on \((\Omega, \mathcal{F}_t, P)\), where \(\mathcal{F}_t\) denotes the enlarged filtration generated by \(\{C(t), t \geq 0\}\) and \(\{W^P(t), t \geq 0\}\).

Denote by \(\mu(C(t)) = \langle \mu, C(t) \rangle\) and \(\sigma(C(t)) = \langle \sigma, C(t) \rangle\), the inner product of vectors, where \(\mu = (\mu_1, ..., \mu_N)'\); denotes a vector of drift parameters \(\mu_i \geq 0\) of the underlying process and \(\sigma = (\sigma_1, ..., \sigma_N)'\); is a vector denoting the volatility parameters \(\sigma_i \geq 0\) associated to each state \(i = 1, ..., N\).

Let \(Z(t)\) denote the logarithmic return on the assets over the time period \([0, t]\),

\[
Z(t) = \ln\left(\frac{S(t)}{S(0)}\right).
\]

Applying Itô’s formula,

\[
Z(t) = \int_0^t \left[ \mu(C(s)) - \frac{1}{2} \sigma^2(C(s)) \right] ds + \int_0^t \sigma(C(s)) dW^P(s) \tag{7.3}
\]

and the process \(S(t)\) may be written as

\[
S(t) = S(0) e^{Z(t)}
\] (7.4)

In terms of occupational times, the first integral on equation 7.3 may be expressed as

\[
\int_0^t \left[ \mu(C(s)) - \frac{1}{2} \sigma^2(C(s)) \right] ds = \sum_{i=1}^N \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) T_i(t) = \overline{T}_\mu(t) - \frac{1}{2} \overline{T}_{\sigma^2}(t)
\]
while the second integral, denoted by \( \phi(t) \) follows a Normal distribution with mean zero and variance

\[
\text{Var} \left( \int_0^t \sigma(C(s)) \, dW(s) \right) = \int_0^t \sigma^2(C(s)) \, ds = \sum_{i=1}^N \sigma_i^2 \bar{T}_i(t) = \bar{T}_{\sigma^2}(t)
\]

Therefore the process \( Z(t) \) may be written as

\[
Z(t) = \bar{T}_\mu(t) - \frac{1}{2} \bar{T}_{\sigma^2}(t) + \phi(t)
\]

and the following characterization is derived.

**Proposition 9** Assume that the risky process \( S(t) \) follows a regime switching dynamic described by equation ?? . Conditioning to the sigma algebra generated by \( \{C(t), t \geq 0\} \), the process \( Z(t) \) is Normal distributed with mean \( M = \bar{T}_\mu(t) - \frac{1}{2} \bar{T}_{\sigma^2}(t) \) and variance \( V = \frac{1}{2} \bar{T}_{\sigma^2}(t) \).

**Lemma 10** The moment generating function of the process \( Z(t) \) is given by

\[
G_{Z(t)}(\theta) = \mathbb{E} \left[ e^{\theta Z(t)} \right] = e^{[\bar{T}_\mu(t) - \frac{1}{2} \bar{T}_{\sigma^2}(t)]\theta + \frac{1}{2} \bar{T}_{\sigma^2}(t)\theta^2}
\]

where

\[
\bar{T}_\mu(t) = \sum_{i=1}^N \mu_i \bar{T}_i(t) ; \quad \bar{T}_{\sigma^2}(t) = \sum_{i=1}^N \sigma_i^2 \bar{T}_i(t)
\]

and \( \bar{T}_i(t) \) denotes the amount of time the Markov chain \( \{C(t)\} \) spends on state \( i \) over the time interval \([0, t]\).

For example, Figure 7.1 illustrates the firm’s assets’ value under a regime switching process with two states. State 1 is characterized by a low or standard volatility while state 2 presents a high volatility behavior. The green lines represent the total time spent on state 1 over the period \([0, T]\), while the red lines represent the total time spent on state 2.
Figure 7.1: The asset’s value of a firm under a volatility regime switching process with two states. Dotted lines represent the time spend on each state.
Chapter 8

Pricing under a regime switching process

Given the switching dynamics of the risk-free asset and the risky asset, the market is incomplete due to the extra uncertainty introduced by changes of regimes\textsuperscript{1}.

Although a risk-neutral probability measure exists if and only if the absence of arbitrage is guaranteed, it is not unique due to the incompleteness of the market. See Harrison and Kreps (1979) and Harrison and Pliska (1981), (1983). Therefore, there are many possibilities to find or construct a probability measure that is risk-neutral\textsuperscript{2}.

Under the regime switching process, Di Masi et al. (1994) take an approach based on the idea of hedging under a mean-variance criterion. Guo (2001) completes the market by introducing an artificial contract that pays one unit when the state switches from one state to another, but as she noted, the existence of such contract is a strong assumption. Elliott

\textsuperscript{1}See Appendix B

\textsuperscript{2}For an introduction to pricing contingent claims in incomplete markets, see Karatzas and Shreve (1991) or Cont and Tankov (2004).
et al. (2005) provide an equivalent martingale measure by using the Esscher transform as in Gerber and Shiu (1994). Finally, Boyle and Draviam (2007) derive the PDE that governs the dynamics of the option under the regime switching process. However as they say on page 269, “we assume that the volatility risk is not priced”. A comparison of these methods is provided in the Appendix.

In this section, we define a risk-neutral probability measure $Q$ such that the expected discounted value of the process $S(t)$ is a martingale, conditioning on the sigma algebra generated by $\{C(t), t \geq 0\}$. Then, the probability measure $Q$ must satisfy the martingale condition

$$
E^Q \left[ e^{-\left[ \tilde{T}_r(t) - \tilde{T}_r(u) \right]} S(t) | S(u) ; u < t ; \mathcal{F}_t^C \right] = S(u) \quad (8.1)
$$

Note that conditioning on the filtration $\mathcal{F}_t^C$ means that the information related to the hidden Markov chain $\{C(t), t \geq 0\}$ is observed by all market participants at time $t$. This is equivalent to say that the market participants, although they do not observe the current state of the economy, they may infer from macro economic time series, the previous states of the economy.

Given the dynamics of the stochastic process $S(t)$, equation 8.1 becomes

$$
E^Q \left[ e^{-\left[ \tilde{T}_r(t) - \tilde{T}_r(u) \right]} S(t) | S(u) ; u < t ; \mathcal{F}_t^C \right] = E^Q \left[ e^{-\left[ \tilde{T}_r(t) - \tilde{T}_r(u) \right]} S(u) e^{\int_u^t \left[ \mu(C(s)) - \frac{1}{2} \sigma^2(C(s)) \right] ds + \int_u^t \sigma(C(s)) dW_P(s)} | S(u) ; u < t ; \mathcal{F}_t^C \right]
$$
\[
\begin{align*}
&= e^{-[\bar{T}_r(t)-\bar{T}_r(u)]} \mathbb{E}_Q \left[ e^{\int_u^t [\mu(C(s)) - \frac{1}{2} \sigma^2(C(s))] ds + \int_u^t \sigma(C(s)) dW^P(s) } \right] \\
&= e^{-[\bar{T}_r(t)-\bar{T}_r(u)]} S(u) \mathbb{E}_Q \left[ e^{\int_{\bar{T}_r(u)}^{\bar{T}_r(t)} \left[ \bar{T}_{\sigma^2}(t) - \frac{1}{2} \bar{T}_{\sigma^2}(u) \right] dt } \right] \\
&= S(u) e^{-[\bar{T}_r(t)-\bar{T}_r(u)]} \mathbb{E}_Q \left[ \bar{T}_{\sigma^2}(t) - \frac{1}{2} \bar{T}_{\sigma^2}(u) \right] \\
\end{align*}
\]

Then, imposing \( \bar{T}_\mu(u) = \bar{T}_r(u) \), for all \( 0 < u < t \), the martingale condition of equation 8.1 is satisfied and the dynamics of the underlying switching process may be written as

\[
dS(t) = r(C(t)) S(t) \, dt + \sigma(C(t)) S(t) \, dW(t)
\]

where \( W(t) \) is a \( Q \)-standard Brownian motion.

The risk-neutral probability measure \( Q \) is Gaussian with mean \( \bar{T}_r(t) - \frac{1}{2} \bar{T}_{\sigma^2}(t) \) and variance \( \bar{T}_{\sigma^2}(t) \), hence

\[
dQ = \frac{1}{\sqrt{2\pi \bar{T}_{\sigma^2}(t)}} e^{-\frac{1}{2 \bar{T}_{\sigma^2}(t)} \left[ Z(t) - (\bar{T}_r(t) - \frac{1}{2} \bar{T}_{\sigma^2}(t)) \right]^2} \, dZ(t)
\]

**European call option**

The following proposition is obtained.

**Proposition 11** Under the regime switching process and conditioning on the whole trajectory of the process \( \{C(t); 0 \leq t \leq T\} \), the weighted European call option pricing formula is given by

\[
C(0, T, S, K, N) = S(0) \mathcal{N}(d_1) - Ke^{-\bar{T}_r(T)} \mathcal{N}(d_2)
\]

where

\[
\begin{align*}
d_1 &= \frac{\ln(S(0)/K) + \bar{T}_r(T) + \frac{1}{2} \bar{T}_{\sigma^2}(T)}{\sqrt{\bar{T}_{\sigma^2}(T)}} \\
d_2 &= \frac{\ln(S(0)/K) + \bar{T}_r(T) - \frac{1}{2} \bar{T}_{\sigma^2}(T)}{\sqrt{\bar{T}_{\sigma^2}(T)}}
\end{align*}
\]
\[ \bar{T}_r (T) = \sum_{i=1}^{n} r_i T_i (T); \quad \bar{T}_{\sigma^2} (T) = \sum_{i=1}^{n} \sigma^2_i T_i (T) \]

\( \mathcal{N} \) is the Gaussian cumulative distribution function and \( T_i (T) \) denotes the amount of time the continuous-time Markov chain spends on state \( i \) over the time period interval \([0, T]\), with \( T \) being the maturity time of the option.

**Proof.** See Appendix A. ■

Some remarks:

**Remark 12** For \( N = 1 \), that is, when there is a single state on the economy, the standard Black and Scholes formula is recovered.

**Remark 13** Under the regime switching process, the price of a European call option becomes a weighted Black and Scholes formula where the parameters of the underlying process and the risk free asset are weighted according to the time spent on each state during the whole period \([0, T]\).

**Remark 14** When the number of states is two, the distribution of the random variable \( T_1 (T) \) is known and it involves a Bessel distribution. (See example provided in Section 2). Applying Theorem 1 in Section 2, the closed-form solution of Guo (2001) is obtained.

**Remark 15** The weighted option pricing formula was obtained by conditioning on the whole trajectory of the process \{\( C (t) ; 0 \leq t \leq T \}\), hence the extra risk due to the switching process was not priced. However, if the joint density function for the occupational times is known, the unconditional option price may be recover as

\[
\int_0^T \ldots \int_0^T C (0, T, S, K, N | \{C (t)\}) \psi (t_1 (t), ..., t_N (t)) d_{t_1 (t)}, ..., d_{t_N (t)}
\]

where \( \psi (t_1 (t), ..., t_N (t)) \) is the joint density function of the occupational times \( T_1, ..., T_N \).
8.1 Numerical procedure

When the number of states equals two, a closed form pricing formula is obtained. However, when the number of states increase, the inversion of the moment generating function of \( \psi(t_1(t), \ldots, t_N(t)) \) is difficult to obtain.

In this section we provide a pseudo-closed form solution to price European call options under a regime switching process with the following algorithm:

1. Discretize the time interval \([0, T]\) on \(m\) intervals of size \(\Delta = \frac{T}{m}\). Note that \(m\) must be large enough such that each \(\Delta_j, j = 1, \ldots, m\) allows at most one switch.

2. Set the number of simulations to be equal to \(l\).

3. Given the matrix \(A(t)\) of the Markov process, simulate a single path of the process \(\{C_j(t); 0 \leq t \leq T; j = 1, \ldots m\}\) which takes value on the \(i = 1, \ldots, N\) possible states.

4. Compute the occupational time \(T_i(T)\) for each state \(i = 1, \ldots, N - 1\). Of course \(T_N(T) = T - \sum_{i=1}^{N-1} T_i(T)\).

5. Compute \(\bar{T}_r(T) = \sum_{i=1}^{N} r_i T_i(T)\) and \(\bar{T}_{\sigma^2}(T) = \sum_{i=1}^{N} \sigma_i^2 T_i(T)\).

6. Apply the weighted Black and Scholes formula.

7. Repeat steps 3 to 6 for \(l\) times in order to get the average of the weighted Black and Scholes formula.

A numerical example is show in Figure 8.1. For simplicity, only the volatility switches. Given the transition probabilities \(p_{11} = 0.9\) and \(p_{22} = 0.15\) and assuming the initial state \(C(0) = 1\), the occupational times are computed. Prices of European call and
Figure 8.1: Simulation of a volatility regime switching process with 2 states.

Put options are calculated for each state separately (no switching) and when the volatility switches. The initial values for the option are: initial stock price $S(0) = 100$; strike price $K = 70$; maturity time $T = 30$ years and $\sigma_1 = 8\%$, $\sigma_2 = 25\%$.

As expected, the paths exhibit volatility clustering and persistence: periods with low volatility are followed by periods with low volatility, then an abrupt change occurs. Note that a two-state volatility switching model exhibits already nice properties similar to the stylized facts documented on empirical financial literature. Even more, a two-state volatility switching model is much easier to understand and implement than others stochastic volatility models.
Chapter 9

Default risk, liquidation and

Parisian options

A debtor that experiences a financial distress can either liquidate his assets or renegotiate its debt. In corporate debt, when the firm’s assets’ value falls below a certain level, debt holders have the right to force liquidation. However, under court protection, some bankruptcy procedures, as the Chapter 11 of the U.S. bankruptcy code, may allow the firm to continue for a fixed grace period in order to prevent immediate liquidation. During such period the firm is under observation: if the firm recovers, it may continue its operations but, if the firm does not show signs of recovery, it may be liquidated definitely.

In order to consider the grace period, liquidation is modelled by introducing a distress clock which accounts for the time the firm’s assets’ value has been below a given default level. Each time the firm recovers, the distress clock is reset to zero. However, the first time the distress clock exceeds the grace period, the firm is liquidated.
9.1 Liquidation

Assume that at time zero, the firm is financed by equity and debt with face value at maturity time $K(T) = K(0)e^{\rho T}$, where $\rho$ is a fixed interest rate, and that the firm can neither issue new debt nor equity during that period.

In order to take into account the grace period, liquidation time is defined as the first time the firm’s assets’ value has spend more than $D$ consecutive units of time below a given default level $L$, which may be written as

$$\tau \equiv \inf \{t > 0 \mid (t - g^S_t) \geq D; S(t) \leq L(t)\}$$

where $g^S_t \equiv \sup \{s \leq t \mid S(s) = L(s)\}$ denotes the last time before $t$ the firm’s assets’ value reaches the default level.

Some remarks:

**Remark 16** When $D = 0$, liquidation arrives as soon as the underlying process hits the barrier for the first time. This case reduces to the first passage time model of Cox 1979.

**Remark 17** The default level $L(t)$ may be interpreted as a barrier that aims to warning the debt holder when the firm is in distress: if the default level is fixed above the debt amount $K(t)$, then the regulator follows strictly control over the performance of the firm. In that case, if the firm is liquidated, it may still cover some payment to the debtors and the liquidation cost.

**Remark 18** The severity of the regulator is captured also on the grace period: the regulator decides the amount of time a firm in distress is allowed to continue doing business before bankrupt definitively.
Without loss of generalization and for simplicity, let the default barrier $L(t)$ be the same as the amount debt $K(t)$ over the period $[0,T]$. Then $L(t) = K(0) e^{\rho t}$, where $K(0)$ and $\rho > 0$ are given constants.

### 9.2 Payoff

We assume that debt holders have priority over equity holders.

**Debt holder**

At maturity time, the debt holder will receive the promised payment minus the eventual loss if the firm is not able to meet its obligations. However, if the firm is liquidated before maturity time, the debt holder, which have priority over equity holders, will receive the promised amount evaluated at liquidation time if the firm’s assets’ value are enough to cover such amount.

**Equity holder**

At maturity time, the equity value is the firm’s asset value minus the promised payment to debt holders. If the value of the assets is not enough to honor its obligations, equity holders receive nothing because debt holders have priority over equity holders. If the firm is liquidated before maturity, equity holders receive the remaining value of the assets, if there is any, after paying the debt value. The payoff is summarized on the Tables 9.1 and 9.2.
9.3 Valuation of debt and equity

The evaluation of the payoff described above may be decomposed in two parts: one part to be paid if default does not arrive up to maturity time, and the second, to be paid if the firm is liquidated at time \( \tau \in (0, T) \).

Equations 9.2 and 9.3 provide the value of debt and equity respectively, under the risk-neutral probability measure \( Q \).

\[
V_D (0, T, S, K, D, N) = e^{-\bar{r}T} \mathbb{E}^Q \left[ \Upsilon_D (S (T)) 1_{\tau \geq T} \right] + \mathbb{E}^Q \left[ e^{-\bar{r}\tau} \Upsilon_D (S (\tau)) 1_{\tau < T} \right]
\]

\[
V_E (0, T, S, K, D, N) = e^{-\bar{r}T} \mathbb{E}^Q \left[ \Upsilon_E (S (T)) 1_{\tau \geq T} \right] + \mathbb{E}^Q \left[ e^{-\bar{r}\tau} \Upsilon_E (S (\tau)) 1_{\tau < T} \right]
\]

Table 9.1: Payoff at maturity date.

<table>
<thead>
<tr>
<th>Assets value</th>
<th>Payoff at maturity time ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S (T) )</td>
<td>( \text{Debt holder} ) ( \min { K (T), S (T) } = K (T) - [K (T) - S (T)]^+ \equiv \Upsilon_D (S (T)) )</td>
</tr>
<tr>
<td>( \text{Equity holder} ) ( \max { 0, S (T) - K (T) } = [S (T) - K (T)]^+ \equiv \Upsilon_E (S (T)) )</td>
<td></td>
</tr>
</tbody>
</table>

Table 9.2: Payoff at liquidation.

<table>
<thead>
<tr>
<th>Assets value</th>
<th>Payoff at liquidation time ( \tau; 0 &lt; \tau &lt; T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S (\tau) )</td>
<td>( \text{Debt holder} ) ( \min { K (\tau), S (\tau) } = K (\tau) - [K (\tau) - S (\tau)]^+ \equiv \Upsilon_D (S (\tau)) )</td>
</tr>
<tr>
<td>( \text{Equity holder} ) ( \max { 0, S (\tau) - K (\tau) } = [S (\tau) - K (0) e^{\rho \tau}]^+ \equiv \Upsilon_E (S (\tau)) )</td>
<td></td>
</tr>
</tbody>
</table>
To solve the above equations, one needs to know the law of the liquidation time \( \tau \) and the value of the firm’s asset at this time, \( S(\tau) \), where liquidation time \( \tau \) is defined as

\[
\tau \equiv \inf \{ t > 0 \mid (t - g^S) \geq D; S(t) \leq L(t) \} \tag{9.4}
\]

and \( g^S \equiv \sup \{ s \leq t \mid S(s) = L(s) \} \) denotes the last time before \( t \) the firm’s assets’ value reaches the default level.

If the underlying process follows a geometric Brownian motion, equations 9.2 and 9.3 are related to pricing Parisian options.

### 9.4 On Parisian options

Parisian options, created by Chesney, Jeanblanc and Yor (1997) are extensions on barrier options. Given a threshold level, a Parisian option may be activated (in) or deactivated (out), as soon as the underlying process reaches the barrier by below (down) or by above (up) for a time period longer than a fixed grace period.

For example, a Parisian down-in option means that the option is activated as soon as the underlying process has been strictly below the threshold for a time period exceeding the fixed grace period.

If the activation depends on the total time the underlying spends beyond the threshold, the option is named cumulative Parisian option.

Parisian options have appealing interpretation on default risk. A Parisian down-out option will lost its value (due at maturity) as soon as the underlying process is below a barrier for a consecutive time exceeding the grace period. In other words, the promised payment due at maturity will be lost if the firm defaults before maturity and it does not
recover during the grace period.

Pricing formulas of Parisian options are useful for evaluate the first term of equations 9.2 and equation 9.3 when the underlying process follows a geometric Brownian motion. However, note that the second term of equations 9.2 and 9.3 are related to American options meaning by that the option (Parisian) may be exercised early.

In order to specify the payoff of Parisian options, the following definitions are required.

1. Consider an indicator function that takes value one if the underlying process $S(t)$ falls to a threshold level $L$ and zero otherwise

   $$1_{S(t) \leq L} = \begin{cases} 
   1 & \text{if } S(t) \leq L \\
   0 & \text{otherwise}
   \end{cases}$$
2. Let $g_{L,t}^S(S)$ denote the last time before $t$ the process $S(t)$ hits the barrier $L$

$$g_{L,t}^S(S) \equiv \sup\{u \leq t \mid S(u) = L\}$$

3. The first time, the excursion time $t - g_{L,t}^S(S)$ exceeds the grace period $D$

$$G_{D,L}^-(S) \equiv \inf\{t > 0 \mid (t - g_{L,t}^S(S)) \geq D\}$$

4. A Parisian down-in option is activated the first time the excursion time $t - g_{L,t}^S(S)$

exceeds the grace period. This is represented by

$$G_{D,L}^-(S) \equiv \inf\{t > 0 \mid (t - g_{L,t}^S(S)) 1_{S(t) \leq L} \geq D\}$$  \hspace{1cm} (9.5)

5. A function that indicates if the option is activated before maturity

$$1_{G_{D,L}^-(S) < T} = \begin{cases} 1 & \text{if } G_{D,L}^-(S) < T \\ 0 & \text{otherwise} \end{cases}$$

The price of a standard Parisian down-in put option, denoted by $P_{in}^-$, with initial

value $S(0) = S$, maturity date $T$, exercise price $K$, barrier level $L$, $0 < L < S(0)$ and grace

period $D < T$ is given by

$$P_{in}^-(0, T, S, K, L, D) = e^{-rT} E^Q\left[(K - S(T))^+ 1_{G_{D,L}^-(S) < T} \mid S(0) = S\right]$$  \hspace{1cm} (9.6)

Chesney, Jeanblanc and Yor (1997) provide a pricing formula when the dynamics

of the underlying process $S(t)$ follows

$$dS(t) = rS(t) dt + \sigma S(t) dW(t)$$
where $W(t)$ is a Brownian motion under the risk-neutral probability measure $Q$, for $r$ and $\sigma$ constants.

By denoting $S(0) = S$, $m \equiv \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$ and $b \equiv \frac{1}{\sigma}\ln\left(\frac{L}{S}\right)$ it follows that

$$S(t) = Se^{\sigma(mT+W_t)}$$

They show that if the underlying process $S(t)$ is transformed to a standard Brownian motion starting at zero, the liquidation time $\tau$ may be transformed to another stopping time: the first time a negative Brownian excursion lasts more than $D$, which Laplace transform is easier to derive$^1$.

Applying Girsanov’s theorem, a new probability measure $\tilde{Q}$ which makes $Z(t) = mt + W_t$ a $\tilde{Q}$-Brownian motion is defined and the activation condition on equation (9.5) becomes

$$G_{D,b}^- \equiv \inf\{t \geq 0 \mid (t - g_{b,t}(S))1_{Z(t) < b} \geq D\}$$

(9.7)

where

$$g_{b,t}(S) \equiv \sup\{u \leq t \mid Z(u) = b\}$$

therefore, the price of a standard Parisian down and in put option, given in equation (9.6) becomes

$$P_{in}^-(0, T, S, K, L, D) = e^{-rT}E_{\tilde{Q}}\left[\left(K - S(T)\right)^+ 1_{G_{D,L}(S) < T}\right]$$

$^1$Chesney et al. (1997) derive the Laplace transform of the first time a negative Brownian excursion lasts more than $D$. See the Appendix for some definitions and results related to Brownian motion.
Chesney, Jeanblanc and Yor (1997) show that

For \( K \leq L \)

\[
P_T^- (0, T, S, K, L, D) = e^{-(r + \frac{1}{2} m^2)T} \int_{-\infty}^{b} e^{my} [K - Se^{\sigma y}] h_2 (T, y) \, dy
\]  

\[
(9.8)
\]

For \( K > L \)

\[
P_T^- (0, T, S, K, L, D) = e^{-(r + \frac{1}{2} m^2)T} \int_{-\infty}^{b} e^{my} [K - Se^{\sigma y}] h_2 (T, y) \, dy
\]  

\[
+ \int_{b}^{\beta} e^{my} [K - Se^{\sigma y}] h_1 (T, y) \, dy
\]  

\[
(9.11)
\]

where \( m \equiv \frac{1}{\sigma} (r - \frac{1}{2} \sigma^2) \); \( S (0) \equiv S; \ b \equiv \ln \left( \frac{L}{S} \right); \ \beta \equiv \ln \left( \frac{K}{S} \right) \) and \( h_1 (t, y), h_2 (t, y) \) are functions with complicate analytical forms that are rather described through their Laplace transform in \( T \) denoted by \( \hat{h}_1 (\lambda, y), \hat{h}_2 (\lambda, y) \) respectively. These are given by

\[
\hat{h}_1 (\lambda, y) = \frac{e^{(2b-y)\sqrt{2\lambda}}}{{\sqrt{2\lambda}}} \Psi (z)
\]

\[
\hat{h}_2 (\lambda, y) = \frac{e^{y\sqrt{2\lambda}}}{{\sqrt{2\lambda}}} \Psi (z) + \frac{\sqrt{2\pi D} e^{\lambda D}}{\Psi (z)} \left[ e^{y\sqrt{2\lambda}} (N (u) - N (-z)) - e^{(2b-y)\sqrt{2\lambda}} N (v) \right]
\]

where

\[
\Psi (z) \equiv 1 + z \sqrt{2\pi} e^{\frac{1}{2} z^2} N (z)
\]
\[
z \equiv \sqrt{2\lambda D}
\]
\[
u \equiv -z - \frac{y - b}{\sqrt{D}}
\]
\[
v \equiv -z + \frac{y - b}{\sqrt{D}}
\]
\[
b \equiv \frac{1}{\sigma} \ln \left( \frac{L}{S} \right)
\]

and \(N\) is the Gaussian cumulative distribution function.

**Remark 19** Chesney et al. (1997) provided a closed-form solution for pricing all combinations of Parisian options, the difficulty comes when inverting the Laplace transforms \(\hat{h}_1(\lambda, y), \hat{h}_2(\lambda, y)\).

**Remark 20** Bernard et al. (2005b) provide a new numerical inverse Laplace transform method adapted to the problem.

### 9.5 Switching Parisians

The Parisian option pricing formulas rely on the assumption that the underlying process follows a geometric Brownian motion. Chesney et al. (1997) show that if the underlying process \(S(t)\) is transformed to a standard Brownian motion starting at zero, the liquidation time \(\tau\) may be transformed to another stopping time \(\tau^0\), which is the first time a negative Brownian excursion lasts more than \(D\). Then, they derive the Laplace transform of \(\tau^0\) by knowing the law of the Brownian meander which is related to the left extremity of a Brownian excursion which straddles time \(t\). (see Appendix).

Unfortunately, when the underlying process follows a regime switching process closed-form formulas are not available, as far as I know. The mathematical results of
standard Parisians (negative Brownian excursions and the law of a Brownian meander) are not applicable. Even for a small time interval, there is a positive probability that the process switch.

Although a closed form solution is not available up to my knowledge, Switching Parisians can be evaluated numerically by discretizing the process.
Chapter 10

Numerical evaluation

[*** INSERT: ***NUMERICAL PROCEDURE ]

[***NUMERICAL EXAMPLES]

[ Insert numerical results from Mathematica

Analyze the switching behavior (persistence probabilities) and how liquidation is affected

Insert graphs ]
Chapter 11

Conclusions

This paper develops a theoretical framework to model default risk by taking into account both: 1) a grace period allowed by an authority (Parisian option) and 2) a regime switching process able to accommodate macro economic variables that may affect the debtor’s payment capacity process. The pricing equations are derived in terms of the moment generating functions of the time spent in each state of the Markov chain. This methodology allows to obtain a pseudo-closed form pricing formula that is easier to compute compared to pure Monte Carlo simulation.

***TO COMPLETE
Bibliography


Part IV

The market value of life insurance liabilities under a regime switching process
Abstract

This paper studies the values of the equity and liabilities of life insurance companies in the presence of regime switching in the economy. Following the contingent claim framework of Grosen and Jørgensen (2002), where the equity and liability of a life insurance company are evaluated in a barrier option framework, this paper proposes a model where the dynamic evolution of the assets follows a geometric Brownian motion with volatility parameter switching according to a continuous-time Markov chain process with discrete state values. After deriving valuation formulas, numerical implementation is illustrated using US life insurance data, providing strong evidence of switching behaviour on the market affecting the contingent claim valuation.

Modern life insurance contracts offer to the policy holder several benefits as, an interest rate guaranteed at the end of the period over the initial amount, or a bonus to be paid in case the insurance company shows high returns, for instance. Other contracts are linked to equity or have a participating policy. However, these contracts have a common risk: if the insurance company defaults and is liquidated, policy holders may not be able to collect the promised benefits. Therefore, it is crucial to consider the risk of default of the insurance company when evaluating the market value of policy contracts. Briys and de Varenne (1997) provide an option pricing framework to study the fair value of life insurance contracts. Following Merton’s approach and assuming life insurance contracts are traded on a liquid and complete market in an arbitrage free world, closed-form formulas are derived. However, the risk of default is evaluated only at maturity date as an European option.

Grosen and Jørgensen (2002) introduce an exponential barrier in order to monitor the risk of default before maturity date. The value of a life insurance contract with a guarantee payment is decomposed in four terms: the final guarantee payment; a European put option related to the risk of default; a bonus European call option if the insurance
assets’ value is in surplus at maturity; and a rebate to be paid early in the case the firm defaults before the maturity date. In this model, the option components are priced under the assumption that the risk-free interest rate is constant and that the volatility of the life insurance assets’ price is constant.

Bernard et al. (2005) propose a valuation of life insurance contracts that takes into account the risk of default in the presence of a stochastic term structure of interest rates. Again, they assume that the life insurance assets’ price follows a geometric Brownian motion with constant volatility.

Chen and Suchanecki (2007) extended the work of Grosen and Jørgensen (2002) by distinguishing default from liquidation as in the bankruptcy framework of François and Morellec (2004). In this setting, the firm is in default when the barrier is reached, then a clock starts ticking and counting the time the firm is in distress. If the firm does not show signs of recovery during a fixed “grace period”, then the firm is liquidated, otherwise, the clock is reset to zero and the firm continues doing business. This model also relies also on the assumption that the risk-free interest rate is constant and the volatility of the life insurance assets’ price is constant.

Since life insurance contracts are mostly written on a long term basis (between 20 to 60 years), it is important to capture the structural changes that the firm’s asset prices may suffer during long periods. Note for instance that during economic recession periods, financial market are characterized by low returns while during expansion periods they exhibit high returns. During stable financial periods, the financial market shows lower volatility while during booms periods the volatility increase excessively.
This paper proposes a regime switching model in order to capture such structural changes. Regime switching models were introduced by Hamilton (1989) to financial econometrics and economists have, since then, found several applications. For example, Garcia and Perron (1996) applied it to model real interest rates and Bansal and Zhou (2002) use it to model the term structure of interest rates. Hardy (2001) considers a regime switching process to model the returns of stocks in the long term.

The goal of this paper is to study the valuation of life insurance contracts in the presence of a switching regime process. Following the framework of Grosen and Jørgensen (2002), it proposes a valuation of life insurance contracts that takes into account, in a simple way, the macroeconomic conditions that may affect the firm’s assets’ value on a long term.

This paper is structured as follows: Section 2 provides the life insurance contract specifications following the framework of Grosen and Jørgensen (2002). Section 3 describes the model to price the liabilities of a life insurance company in the presence of a switching regime process. Section 4 describe the econometric procedure to estimate the model parameters. Then, an empirical evaluation to seven US life insurance companies is performed. Finally Section 6 concludes.
Chapter 12

Contract specifications

Following the framework of Grosen and Jørgensen (2002), which work was inspired by Briys and de Varenne (1997), the policy holder and the equity holder agree on establish a life insurance company with an initial assets’ value $S(0)$. The representative policy holder (also named the liability holder) participates with a premium payment $L(0) = \alpha S(0)$, $\alpha \in (0,1)$, which constitutes the liability of the insurance company at time zero. On the other hand, the representative equity holder participates with $E(0) = (1 - \alpha) S(0)$, which is the value of equity at time zero. The capital structure of the insurance company at time zero is presented on the following balance sheet.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(0)$</td>
<td>$E(0) = (1 - \alpha) S(0)$</td>
</tr>
<tr>
<td></td>
<td>$L(0) = \alpha S(0)$</td>
</tr>
<tr>
<td>$S(0)$</td>
<td>$S(0)$</td>
</tr>
</tbody>
</table>

12.1 The payoff at maturity time

The claim of liability’s holders
Most life insurance policies promise the policy holder a continuously compounded return on its initial contribution to be paid at the end of the contract. This promised payment may be written as $L(T) = L(0)e^{r_g T}$, where $r_g$ is the minimum guaranteed interest rate fixed at the beginning of the contract. Such promise can be honored only if, at maturity date, the insurance company assets’ value is enough to cover the promised amount. Otherwise, since policy holders have priority over equity holders, the policy holder will receive the firm’s assets’ value.

In addition to the guaranteed payment, the policy holder is entitled to receive a bonus in case the insurance company has a surplus at maturity. Denote by $\delta \in [0,1]$ the participation rate for the policy holder.

The claim of the policy holder at maturity date is

$$\Upsilon_L (S(T)) = L(T) - [L(T) - S(T)]^+ + \delta [\alpha S(T) - L(T)]^+$$ (12.1)

This payoff may be decomposed on three parts:

1. A deterministic guaranteed payment due at maturity $L(T) = L(0)e^{r_g T}$,

2. A European put option payoff $[L(T) - S(T)]^+$, resulting from the fact that the equity holder has limited liability and

3. A European call option payoff $\delta [\alpha S(T) - L(T)]^+$, corresponding to the bonus in case the insurance company has a surplus.

The underlying process of the option is the firm’s assets’ value $\{S(t)\}; t \in [0,T]$, while the strike price at maturity is the guaranteed payment $L(T)$.

**The claim of equity holders**
As residual claimants, the value of equity at maturity date consists on the firm’s assets’ value minus the payment provided to policy holders:

\[ \Upsilon_E (S (T)) = [S (T) - L (T)]^+ - \delta [\alpha S (T) - L (T)]^+ \] (12.2)

This payoff consists on the difference of two European call options on the firm’s assets’ value \( S (T) \) with strike price being the guaranteed payment \( L (T) \).

The following table summarizes the payoff at maturity time.

<table>
<thead>
<tr>
<th></th>
<th>( S (T) )</th>
<th>( L (T) ) - ( L (T) - S (T) )(^+) + \delta [\alpha S (T) - L (T)]^+</th>
<th>( \Upsilon_L (S (T)) )</th>
<th>( \Upsilon_E (S (T)) ) = ([S (T) - L (T)]^+ - \delta [\alpha S (T) - L (T)]^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability</td>
<td>( \Upsilon_L (S (T)) )</td>
<td>( L (T) - [L (T) - S (T)]^+ )</td>
<td>Equity holder</td>
<td>( \Upsilon_E (S (T)) ) = ([S (T) - L (T)]^+ - \delta [\alpha S (T) - L (T)]^+)</td>
</tr>
<tr>
<td></td>
<td>( S (T) )</td>
<td>( \Upsilon_E (S (T)) ) = ([S (T) - L (T)]^+ - \delta [\alpha S (T) - L (T)]^+)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 12.1: payoff at maturity date

12.2 The payoff at liquidation

Insurance companies may be in distress before maturity and consequently not be able to meet its obligations when due. Therefore, the risk of default before maturity must be considered when modelling the market value of liabilities.

Grosen and Jørgensen (2002) introduce a barrier in order to monitor the risk of default during the life of the contract. Liquidation is declared as soon as the insurance company assets’ value is below a given barrier:

\[ \tau = \inf \{ t \in [0, T] \mid S (t) \leq B (t) \} \] (12.3)
where the barrier is defined as

\[ B(t) \equiv \lambda L(0) e^{r \tau^t}; \ t \in [0, T) \]  

for some specified constant \( \lambda \).

If \( \lambda \geq 1 \) and the barrier has been reached before maturity date, the insurance company is in default but still it is able to repay the policy holder his initial deposit compounded at the guaranteed interest rate up to liquidation time. In this case, equity holders have a surplus \( S(\tau) - L(0) e^{r \tau^t} = (\lambda - 1) L(0) e^{r \tau^t} \), which may be used to pay legal or administrative fees, so this term is not priced on the model.

If \( \lambda < 1 \) and the barrier has been reached before maturity date, the insurance company is not able to meet its obligations. Therefore, due to the priority rule, the liability holder will receive the entire value of the assets at liquidation time while equity holders receive nothing.

The payoff at liquidation is summarized below:

<table>
<thead>
<tr>
<th>Assets' value</th>
<th>( S(\tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liability holder</td>
<td>( \Upsilon_L(S(\tau)) = \min{\lambda, 1} L(0) e^{r \tau^t} )</td>
</tr>
<tr>
<td>Equity holder</td>
<td>( \Upsilon_E(S(\tau)) = 0 )</td>
</tr>
</tbody>
</table>

Table 12.2: Payoff at liquidation time \( \tau \)

The fair valuation of contingent claims are given by the expected discounted payoff under the risk-neutral probability measure \( Q \)

The fair value of liabilities of an insurance company becomes
\[ V_L (0, T, S, L) = \mathbb{E}^Q \left[ e^{-rT} \Upsilon_L (S (T)) 1_{\tau \geq T} \right] + \mathbb{E}^Q \left[ e^{-\tau r} \Upsilon_L (S (\tau)) 1_{\tau < T} \right] \quad (12.5) \]

while the value of equity is given by

\[ V_E (0, T, S, L) = \mathbb{E}^Q \left[ e^{-rT} \Upsilon_E (S (T)) 1_{\tau \geq T} \right] \quad (12.6) \]

where \( r \) denotes a risk free interest rate available on the market.

Grosen and Jørgensen (2002) presented solutions for the above equations under the assumption that the dynamics of the firm’s assets’ value follows a geometric Brownian motion with constant volatility during the life of the contract. In order to accommodate the macro economic fluctuations that may affect the dynamics of the firm’s assets’ value, this paper consider a regime switching model, which is described in the following section.
Chapter 13

Model

13.1 Switching regime framework

Let \((\Omega, \mathcal{F}_t, P)\), be a complete probability space, where \(P\) is the real world or physical probability measure.

Assume that the dynamics of the firm’s assets’ value \(\{S(t), t \geq 0\}\) is described by the following process

\[
dS(t) = \mu(C(t)) S(t) \, dt + \sigma(C(t)) S(t) \, dW^P(t)
\]  

(13.1)

where \(\mu(C(t)) = \langle \mu, C(t) \rangle\) and \(\sigma(C(t)) = \langle \sigma, C(t) \rangle\) are the drift and the volatility parameters respectively and \(W^P(t)\) is a standard Brownian motion on \((\Omega, \mathcal{F}_t, P)\).

In this paper, instead to assume the parameters are constant over time, we allow them to switch from different values according to the prevalent state of the economy. More precisely, we suppose that the switching of the parameters are driven by a continuous-time homogeneous Markov chain \(\{C(t); t \geq 0\}\) defined on a probability space \((\Omega, \mathcal{F}^C_t, P)\).
The discrete state space of $N$ elements is represented by $E = \{e_1, ..., e_N\}$ where $e_i = [0, ..., 0, 1, 0, ... 0]', \ i = 1, ..., N \in \mathbb{N}$ is the indicator vector having one at the $i$-th element and zeros elsewhere, and the prime $'$ denotes transpose. $\mathcal{F}^C_t = \sigma(\{C(t)\})$ denotes the complete filtration generated by the process $\{C(t), t \geq 0\}$.

Let $N$ be the number of states of the economy. Denote by $\mu = (\mu_1, ..., \mu_N)'$ the vector of drift parameters $\mu_i$ of the firm’s assets’ value and by $\sigma = (\sigma_1, ..., \sigma_N)'$ the vector of volatility parameters $\sigma_i$ of the firm’s assets’ value associated to each state $i = 1, ..., N$. Thus, $\mu(C(t)) = \langle \mu, C(t) \rangle$ and $\sigma(C(t)) = \langle \sigma, C(t) \rangle$ represent the inner product of vectors.

We assume that the continuous-time Markov chain $\{C(t); t \geq 0\}$, defined on the filtration $\mathcal{F}^C_t$ is independent of $W^P(t)$ a standard Brownian motion on $(\Omega, \mathcal{F}_t, P)$, and we denote by $\mathcal{F}_t$ the enlarged filtration generated by $\{C(t), t \geq 0\}$ and $\{W^P(t), t \geq 0\}$.

Suppose that the market contains a riskless asset $B^0(t)$ whose risk-free interest rate evolves according to the same continuous-time Markov chain $\{C(t); t \geq 0\}$. Let $r(C(t)) = \langle r, C(t) \rangle; \ r = (r_1, ..., r_N)'$; $r_i > 0$ be the vector of instantaneous risk-free rates associated to each possible state $i = 1, ..., N$. The dynamics of the risk-free asset is described by

$$dB^0(t) = r(C(t)) B^0(t) dt \quad (13.2)$$

Let $T_i(t)$ denote the amount of time the continuous-time Markov chain spends on state $i$ over the time interval $[0, t]$, which may be expressed as an indicator function or inner product of vectors by

$$T_i(t) = \int_0^t 1_{C(s)=e_i} ds \ \text{or} \ T_i(t) = \int_0^t \langle e_i, C(s) \rangle ds \quad (13.3)$$

therefore, the solution of equation 13.2 is

$$B^0(t) = B^0(0) e^{Tr(t)}$$
where $\bar{T}_r(t) \equiv \sum_{i=1}^{N} r_i T_i(t)$ represents the weighted sum of the times spent on the states $i$, $T_i(t)$, $i = 1, \ldots, N$, over the interval $[0, t]$ with weights equal to the instantaneous risk-free interest rate $r_i$.

If the market consists on two instruments only: the risky firm’s assets and the riskless asset, then the dynamics described on equation 13.1 leads to an incomplete market because there is an extra risk introduced by the switching behavior that is not hedged only with the two financial instruments.

Assuming there is no arbitrage, a risk-neutral probability measure exits but is not unique. By imposing a martingale condition, a risk-neutral probability measure is derived. Hence, conditioning on the process $\{C(t); t \geq 0\}$, the dynamics of the underlying process may be written as

$$dS(t) = r(C(t)) S(t) dt + \sigma(C(t)) S(t) dW(t)$$

(13.4)

where $W(t)$ is a $Q$-standard Brownian motion.

Let $Z(t) = \ln \left( \frac{S(t)}{S(0)} \right)$, then

$$Z(t) = \int_{0}^{t} \left[ r(C(s)) - \frac{1}{2} \sigma^2(C(s)) \right] ds + \int_{0}^{t} \sigma(C(s)) dW(s)$$

(13.5)

By conditioning on the process $\{C(t); t \geq 0\}$ and assuming absence of arbitrage, a risk-neutral probability measure $Q$ is derived, being a Gaussian distribution with mean $M = \bar{T}_r(t) - \frac{1}{2} \bar{T}_{\sigma^2}(t)$ and variance $V = \bar{T}_{\sigma^2}(t)$.

Hence

$$dQ = \frac{1}{\sqrt{2\pi \bar{T}_{\sigma^2}(t)}} e^{-\frac{1}{2\bar{T}_{\sigma^2}(t)}[Z(t)-(\bar{T}_r(t)-\frac{1}{2}\bar{T}_{\sigma^2}(t))]^2} dZ(t)$$

(13.6)
where and \( \bar{T}_{\sigma^2} (t) = \sum_{i=1}^{N} \sigma_i^2 T_i (t) \) is the sum of volatilities weighted by the time spent on each state up to time \( t \).

### 13.2 Valuation

**Market value of liabilities**

The market value of liabilities of an insurance company under the regime switching process is given by

\[
V_L (0, T, S, L, N) = \mathbb{E}^Q [e^{-\bar{T}_r (T)} \gamma_L (S (T)) \mathbf{1}_{\tau \geq T}]
\]

\[
+ \mathbb{E}^Q [e^{-\bar{T}_r (\tau)} \gamma_L (S (\tau)) \mathbf{1}_{\tau < T}]
\]

Note that the first expectation is conditioned to the event that the process does not hit the barrier during the life of the option. The first expectation becomes

\[
\mathbb{E}^Q [e^{-\bar{T}_r (T)} \gamma_L (S (T)) \mathbf{1}_{\tau \geq T}]
\]

\[
= \mathbb{E}^Q [e^{-\bar{T}_r (T)} L (T) \mathbf{1}_{\tau \geq T}] - \mathbb{E}^Q [e^{-\bar{T}_r (T)} [L (T) - S (T)]^+ \mathbf{1}_{\tau \geq T}]
\]

\[
+ \mathbb{E}^Q [\alpha \delta e^{-\bar{T}_r (T)} \left( S (T) - \frac{L (T)}{\alpha} \right)^+ \mathbf{1}_{\tau \geq T}]
\]

The second expectation is conditioning to the event that the process hits the barrier before maturity.

\[
\mathbb{E}^Q [e^{-\bar{T}_r (\tau)} \gamma_L (S (\tau)) \mathbf{1}_{\tau < T}] = \min \{ \lambda, 1 \} L (0) \mathbb{E}^Q [e^{-\bar{T}_r (\tau) + \tau \gamma} \mathbf{1}_{\tau < T}]
\]

Therefore, the market value of liabilities may be written as

\[
V_L (0, T, S, L, N) = G - PO + BO + RB
\]  

(13.7)
where

\[ G \equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} L(T) 1_{\tau \geq T} \right] \] is the guaranteed payment due at maturity,

\[ PO \equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} [L(T) - S(T)]^+ 1_{\tau \geq T} \right] \] is a put option payoff in case of no default,

\[ BO \equiv \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left( S(T) - \frac{L(T)}{\alpha} \right)^+ 1_{\tau \geq T} \right] \] is the bonus call option if there is a surplus on the firm’s assets, and

\[ RB \equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \min \{\lambda, 1\} L(0) e^{\tau_s} 1_{\tau < T} \right] \] is the rebate to be paid in case of early liquidation.

The underlying of the put and call option is the firm’s assets’ value \( S(t) \) and the strike price is the promised payment \( L(T) \) for the put option and \( \frac{L(T)}{\alpha} \) for the bonus option.

**Value of equity**

The value of equity, under the regime switching process becomes

\[
V_E(0, T, S, L, N) = \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \Upsilon_E(S(T)) 1_{\tau \geq T} \right] \\
= \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} [S(T) - L(T)]^+ 1_{\tau \geq T} \right] \\
- \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left( S(T) - \frac{L(T)}{\alpha} \right)^+ 1_{\tau \geq T} \right]
\]

which may be written as

\[
V_E(0, T, S, L, N) = CO - BO
\] (13.8)

where

\[
CO = \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} [S(T) - L(T)]^+ 1_{\tau \geq T} \right] \] is a call option contract in case of no default. The underlying of the option is the firm’s assets’ value \( S(t) \) and the strike price is the promised payment \( L(T) \).

**Fair valuation**
Grosen and Jørgensen (2002) derived closed form solutions for equation 13.7 and equation 13.8 when the underlying process follows a geometric Brownian motion with constant parameters and constant risk-free interest rate. In this paper we assume the dynamics of the firm’s assets’ value is described by equation 13.4. When the evaluation is at maturity time, closed form solutions may be derived for the regime switching underlying process. However, when the evaluation depends on the hitting time $\tau$, we will use numerical procedures for the evaluation.

Equation 13.7 and equation 13.8 may be written as

\[
V_L(0, T, S, L, N) = e^{-\tilde{r}_r(T)L(T)}\left[1 - Q_1 + Q_2 + \delta Q_5\right] - Q_3 - \alpha \delta E_4^Q + \min\{\lambda, 1\} L(0) E_6^Q + \alpha \delta SWCO \left[\frac{L(T)}{\alpha}\right] - SWPO \left[L(T)\right]
\]

\[
V_E(0, T, S, L, N) = e^{-\tilde{r}_r(T)L(T)}[Q_8 - \delta Q_5] + \alpha \delta E_4^Q - E_7^Q + SWCO \left[L(T)\right] - \alpha \delta SWCO \left[\frac{L(T)}{\alpha}\right]
\]

The terms $SWCO[K]$ and $SWPO[K]$ denote a European call and a European put option respectively, where the underlying follows a regime switching process and the strike price is $K$. Under certain assumptions, these options have a closed form formulae (see Monter (2008a)).

Thus, the computation of $V_L$ and $V_E$ relies on 1) closed form solutions when the
options are evaluated at maturity, and 2) the law of the hitting time \( \tau \) under a switching regime process \( Q_1 \equiv Q(\tau < T) \). On the remainder of the paper, we will compute by simulation, the terms related to the hitting time \( \tau \). See the Appendix.
Chapter 14

Empirical study of US life insurance companies

One of the most prolific research fields on empirical finance is the study of assets’ returns volatility\(^1\). Since the seminal ARCH model of Engle (1982) and the GARCH model of Bollerslev (1986), several models have been proposed to capture volatility clustering. However, ARCH models study volatility as a deterministic function of past observations (squares of lagged residuals). For a survey on the literature see Gouriéroux (1997). On the other hand, Stochastic Volatility models (SV) (see Ghysels et al. (1996)) suppose that volatility follows some latent stochastic process. Although these models are more sophisticated than ARCH and GARCH models, they are, in general, more difficult to estimate.

It is well known that market returns exhibit abrupt changes with somewhat persistence over some time periods. Those stylized facts are better captured by switching regime

\(^1\)For an introduction to financial econometric studies, see Campbell et al. (1997) and Gouriéroux and Jasiak (2001).
models. Since the seminal econometric work of Hamilton (1989)\(^2\), switching regime models have been applied extensively on modelling business cycles (Hamilton (1989)), exchange rates (Engel and Hamilton (1990)), real interest rates (Garcia and Perron (1996)), term structure of interest rates (Bansal and Zhou (2002)), stock returns in the long term (Hardy (2001)), among other applications.

This section reviews the methodology to estimate a regime switching model based on the work of Hamilton (1989). Then, the switching regime structure parameters of US real interest rates and Life insurance stock prices.

### 14.1 Econometric specification

The model described in Section 3 results a mixture of Gaussian distributions. We concentrate on estimate only the volatility parameter that switch according to a Markov chain with two possible states. Although more than two states are possible to consider, it does not add value.

For simplicity of notation, let \( C(t) = \{0, 1\} \) representing the two states. Following Hamilton (1994), the model may be written as

\[
R(t) = \mu + [\sigma_0 (1 - C(t)) + \sigma_1 C(t)] \varepsilon(t)
\]  

(14.1)

where \( R(t) \equiv 100 \ln \left( \frac{S(t)}{S(t-1)} \right) \) are the logarithm returns between two observations\(^3\) and \( \varepsilon(t) \) are independent random variables that are Gaussian distributed.

Let \( \{ C(t) ; t \geq 0 \} \) be a two-state homogeneous Markov chain, independent of \( \varepsilon(t) \),

\(^2\)Statistical work was initiated by Quandt (1958).

\(^3\)A known property is that logarithm returns helps the series to be stationary, which is a useful property when estimating the model.
with transition probability matrix

\[ P = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix} \]

where \( p_{00} \equiv \Pr[C(t) = 0 \mid C(t-1) = 0] \) and \( p_{11} \equiv \Pr[C(t) = 1 \mid C(t-1) = 1] \) are the persistence probabilities. Assuming stationarity, the unconditional probabilities are defined as \( \pi_0 \equiv \Pr[C(t) = 0] \) and \( \pi_1 \equiv \Pr[C(t) = 1] \), then

\[
\begin{align*}
\pi_0 &= p_{00}\pi_0 + (1 - p_{11})\pi_1 \\
\pi_1 &= p_{11}\pi_1 + (1 - p_{00})\pi_0
\end{align*}
\]

therefore,

\[
\begin{align*}
\pi_0 &= \frac{1 - p_{11}}{2 - p_{00} - p_{11}}; \\
\pi_1 &= \frac{1 - p_{00}}{2 - p_{00} - p_{11}}
\end{align*}
\] (14.2)

Denote the vector of the observed past returns by

\[ \mathbf{R}(t) = [R(t), R(t-1), ..., R(1)]' \]

and let \( \theta = [\mu, \sigma_0, \sigma_1, \pi_0, \pi_1]' \) be the vector of parameters to be estimated.

The density function of the log returns, conditional to \( \{C(t); t \geq 0\} \) (see equation 13.6), is

\[
f(R(t) \mid C(t); \theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{[R(t) - \mu]^2}{\sigma_0(1 - C(t)) + \sigma_1 C(t)}}
\] (14.3)

Therefore, the likelihood function is

\[
L = f(R(T); \theta) = f(R(T) \mid R(T-1); \theta) f(R(T-1) \mid R(T-2); \theta) ... f(R(2) \mid R(1); \theta)
\]
The maximum likelihood estimate is the parameter vector $\theta$ that maximizes

$$\theta \in \max_{\theta} \ln f(R(T); \theta)$$

then, the likelihood for a given $\theta$ is computed in a recursive fashion applying Bayes’ rule.

**Estimation procedure**

Applying Bayes’ rule

$$f(R(t) | R(t-1); \theta) = \frac{f(R(t); \theta)}{f(R(t-1); \theta)}$$

$$= \frac{1}{f(R(t-1); \theta)} \sum_{C(t)=0}^{1} f(R(t), C(t); \theta)$$

$$= \frac{1}{f(R(t-1); \theta)} \sum_{C(t)=0}^{1} f(R(t) | R(t-1), C(t); \theta) f(R(t-1), C(t); \theta)$$

(14.6)

but

$$\frac{f(R(t) | R(t-1), C(t); \theta)}{f(R(t-1); \theta)} = f(R(t) | C(t); \theta)$$

$$\frac{f(R(t-1), C(t); \theta)}{f(R(t-1); \theta)} = \Pr[C(t) | R(t-1); \theta]$$

then equation 14.6 becomes

$$f(R(t) | R(t-1); \theta) = \sum_{C(t)=0}^{1} f(R(t) | C(t); \theta) \Pr[C(t) | R(t-1); \theta]$$

(14.7)

The first term, $f(R(t) | C(t); \theta)$ is given by equation 14.3, while the second is computed as

$$\Pr[C(t) | R(t-1); \theta]$$

\[
\begin{align*}
&= \Pr\left[ C(t), R(t-1) ; \theta \right] \\
&= \sum_{C(t-1)=0}^{1} \Pr\left[ C(t), C(t-1), R(t-1) ; \theta \right] \\
&= \sum_{C(t-1)=0}^{1} \frac{\Pr\left[ C(t), C(t-1), R(t-1) ; \theta \right]}{f(R(t-1) ; \theta)} \\
&= \sum_{C(t-1)=0}^{1} \frac{\Pr\left[ C(t) \mid C(t-1), R(t-1) ; \theta \right] \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right]}{f(R(t-1) ; \theta)} \\
&= \sum_{C(t-1)=0}^{1} \Pr\left[ C(t) \mid C(t-1) ; \theta \right] \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right] \\
&= \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right]
\end{align*}
\]

Then,
\[
\begin{align*}
&= \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right] \\
&= \sum_{C(t-1)=0}^{1} \frac{\Pr\left[ C(t), C(t-1), R(t-1) ; \theta \right]}{f(R(t-1) ; \theta)} \\
&= \sum_{C(t-1)=0}^{1} \frac{\Pr\left[ C(t) \mid C(t-1), R(t-1) ; \theta \right] \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right]}{f(R(t-1) ; \theta)} \\
&= \sum_{C(t-1)=0}^{1} \frac{\Pr\left[ C(t) \mid C(t-1) ; \theta \right] \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right] \Pr\left[ C(t-1) \mid R(t-2) ; \theta \right]}{f(R(t-1) \mid C(t-1) ; \theta) \Pr\left[ C(t-1) \mid R(t-2) ; \theta \right]} \\
&= \sum_{C(t-1)=0}^{1} \frac{\Pr\left[ C(t) \mid C(t-1) ; \theta \right] \Pr\left[ C(t-1) \mid R(t-1) ; \theta \right]}{f(R(t-1) \mid C(t-1) ; \theta) \Pr\left[ C(t-1) \mid R(t-2) ; \theta \right]}
\end{align*}
\]

Then, $f(R(t-1) \mid C(t-1) ; \theta)$ is given by equation 14.3, while $\Pr\left[ C(t-1) \mid R(t-2) ; \theta \right]$ must be calculated recursively.

Therefore, the maximum likelihood for a given $\theta$ is obtained by iterating equation 14.7, which requires the computation of equation 14.8 that involves 14.9 and so on.

The initial probability $\Pr\left[ C(1) \mid R(0) ; \theta \right]$ can be approximate by the unconditional probabilities which satisfy equation 14.2.
14.2 Data description

The regime switching model will be estimated for the log returns of seven life insurance companies traded on the US market\(^5\). The observation period runs from January 1973 to April 2008; consisting on more than 400 monthly observations\(^6\).

Figure 14.1 illustrates the stock prices of each firm and Figure 14.2 provide the logarithm returns. The graphs suggest volatility clustering, in other words, it seems that observations with low volatility are followed by observations with low volatility (persistence) until the regime switch to high volatility. Figure 14.3 summarizes the descriptive statistics of the monthly return series. Some of them show right skewness while others left. All distributions are leptokurtic (peaked relative to the normal distribution). The Jarque-Bera test for testing normality is rejected for all series. Note that the companies Citizens and Pres exhibit extreme values for short periods, and a switching regime model alone is not convenient. Therefore, we concentrate on the remaining seven insurance firms.

14.3 Empirical results

The filtered probabilities, as described in equations 14.7 to 14.9 are computed for each life insurance company. Figures 14.4, 14.6, 14.8, 14.10, 14.12, 14.14 and 14.16, show the logarithm return of the stock prices. Figures 14.5, 14.7, 14.9, 14.11, 14.13, 14.15 and 14.17 show the estimated volatility of the returns (blue line) and the regime switching process with two states (orange dotted line) of each firm. The estimation of the regime switching

\(^5\)Data source: Datastream, Thompson Financial. Kindly provided by the CEDIF (Centre de Documentation Financière) at University of Lausanne.

\(^6\)A daily data analysis may be added (upon request). Then one should comment about time aggregation differences on the parameter estimation.
Figure 14.1: Stock prices. Monthly data 1973:1 2008:4

Figure 14.2: Logarithm returns of stock prices. Monthly data 1973:1 2008:4
Figure 14.3: Descriptive statistics of monthly returns. Sample: April 1975 to April 2008. 397 observations.

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<thead>
<tr>
<th>Stock</th>
<th>AFLAC</th>
<th>AMERNAT</th>
<th>CITIZENS</th>
<th>KANSAS</th>
<th>LINCOLN</th>
<th>NATWESTERN</th>
<th>PRES</th>
<th>PROTECTIVE</th>
<th>TORCHMARK</th>
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<td>0.68</td>
<td>0.67</td>
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<td>18.82</td>
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Figure 14.4: AFLAC log return of stock prices.

Volatilities are summarized in Figure 14.18. The estimated volatility, their statistics, the transition probabilities and the occupational times are reported.

Figure 14.5: AFLAC estimated volatility and two states regime switching.
Figure 14.6: AMERNAT log return of stock prices.

Figure 14.7: AMERNAT estimated volatility and two states regime switching.

Figure 14.8: KANSAS log return of stock prices.

Figure 14.9: KANSAS estimated volatility and two states regime switching.
Figure 14.10: LINCOLN log return of stock prices.

Figure 14.11: LINCOLN estimated volatility and two states regime switching.

Figure 14.12: NATWESTERN log return of stock prices.

Figure 14.13: NATWESTERN estimated volatility and two states regime switching.
Figure 14.14: PROTECTIVE log return of stock prices.

Figure 14.15: PROTECTIVE estimated volatility and two states regime switching.

Figure 14.16: TORCHMARK log return of stock prices.

Figure 14.17: TORCHMARK estimated volatility and two states regime switching.
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State 1

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<th>skew</th>
<th>exc kurt</th>
<th>min</th>
<th>max</th>
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Figure 14.18: Parameter estimation two states regime switching volatility. Monthly data: April 1975 to April 2008. 396 observations.
Analysis of empirical results

*****[TO COMPLETE]
Chapter 15

Conclusions

*****[TO COMPLETE]
Bibliography


Appendix A

Appendix Part II

A.1 Proof of proposition 2

Before proving Proposition 2, we establish the following well known result:

Lemma : If \( \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \) follows a bivariate normal distribution with

\[
\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \text{Var} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} w_1^2 & \rho w_1 w_2 \\ \rho w_1 w_2 & w_2^2 \end{pmatrix}
\]

then

\[
\mathbb{E} (\exp (Z_1 \cdot 1_{Z_2 \geq 0}) = \exp \left( m_1 + \frac{w_1^2}{2} \right) \Phi \left( \frac{m_2}{w_2} + \rho w_1 \right)
\]

where \( \Phi () \) is the cumulative normal distribution function.

We applied this result to proof proposition 2:

Proof. By Proposition 1, the default risk premium is given by

\[
P_t = K \cdot \mathbb{E}_t \{ H() - G() \},
\]

therefore we just need to compute

\[
H() = \mathbb{E}_t [\exp (A) \cdot 1_{B \geq 0} | U^T] \quad \text{and} \quad G() = \mathbb{E}_t [\exp (A - B) \cdot 1_{B \geq 0} | U^T]
\]
By hypothesis, the conditional probability distribution of the variables \( \log X_{t+1}, \log \frac{D_{t+1}}{D_t} \) given the economy state \( U^\tau \) at time \( \tau = 1, \ldots, T \) is a bivariate normal distribution, therefore the conditional probability distribution of the variables \( (A, B) \) given the economy state \( U^t \) is also a bivariate normal distribution (of course with different parameters), then by applying the Lemma 1 we have

\[
H() = \mathbb{E}_t \left[ \exp \left( A \cdot 1_{B \geq 0} \mid U^T \right) \right] = \exp \left( \mu_A + \frac{1}{2} \sigma_A^2 \right) \Phi \left( \frac{\mu_B}{\sigma_B} + \rho_{AB} \sigma_A \right).
\]

Lets define \( C = A - B \), then the variables \( (A, C) \) given the economy state \( U^t \) follows a bivariate normal distribution (again with different parameters), then by applying the Lemma 1 we have

\[
G() = \mathbb{E}_t \left[ \exp \left( C \cdot 1_{B \geq 0} \mid U^T \right) \right] = \exp \left( \mu_C + \frac{1}{2} \sigma_C^2 \right) \Phi \left( \frac{\mu_B}{\sigma_B} + \rho_{BC} \sigma_C \right)
\]

where \( \mu_c = \mu_A - \mu_B \), \( \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} \), \( \rho_{BC} = \frac{\sigma_{BC}}{\sigma_B \sigma_C} \), \( \sigma_{BC} = \sigma_{AB} - \sigma_B^2 \), \( \sigma_C^2 = \sigma_A^2 + \sigma_B^2 - 2 \sigma_{AB} \), thus

\[
H() = \exp \left( \mu_A + \frac{1}{2} \sigma_A^2 \right) \Phi (d_1)
\]

\[
G() = \exp \left( \mu_A - \mu_B + \frac{1}{2} \left( \sigma_A^2 + \sigma_B^2 - 2 \sigma_{AB} \right) \right) \Phi (d_2) \text{ with }
\]

\[
d_1 = \frac{\mu_B + \sigma_{AB}}{\sigma_B}, \quad d_2 = d_1 - \sigma_B
\]

therefore

\[
P_t = K \cdot \mathbb{E}_t \left[ \exp \left( \mu_A + \frac{1}{2} \sigma_A^2 \right) \Phi (d_1) - \exp \left( \mu_A - \mu_B + \frac{1}{2} \left( \sigma_A^2 + \sigma_B^2 - 2 \sigma_{AB} \right) \right) \Phi (d_2) \right]
\]

\[\blacksquare\]

We denoted by \( X_t \) the stochastic discount factor that will serve to price default risk and by \( D_t \) the underlying debt service process, therefore, by definition of \( A \) and \( B \) we
have:

\[ A = \sum_{\tau=t}^{T-1} \log (m_{\tau,\tau+1}) = \sum_{\tau=t}^{T-1} \log (X_{\tau,\tau+1}) \]

\[ B = \log k - \sum_{\tau=t}^{T-1} \log \left( \frac{S_{\tau+1}}{S_{\tau}} \right) = \log \left( \frac{K}{D_t} \right) - \sum_{\tau=t}^{T-1} \log \left( \frac{D_{\tau+1}}{D_{\tau}} \right) \]

Since we assume that the conditional probability distribution of \((\log X_{t+1}, \log \frac{D_{t+1}}{D_t})\) given the economy state \(U_t\), is a bivariate normal distribution with parameters

\[
\begin{pmatrix}
\log X_{t+1} \\ \log \frac{D_{t+1}}{D_t}
\end{pmatrix} \sim \mathcal{N}
\begin{pmatrix}
\mu_{X_{t+1}} \\ \mu_{D_{t+1}}
\end{pmatrix},
\begin{pmatrix}
\sigma^2_{X_{t+1}} & \sigma_{X D_{t+1}} \\ \sigma_{X D_{t+1}} & \sigma^2_{D_{t+1}}
\end{pmatrix}
\]

then the parameters of the variables \((A, B)\) are

\[
\begin{align*}
\mu_A &= \mathbb{E} (A) = \mathbb{E} \left( \sum_{\tau=t}^{T-1} \log (X_{\tau,\tau+1}) \right) = \sum_{\tau=t}^{T-1} \mathbb{E} (\log (X_{\tau,\tau+1})) = \sum_{\tau=t}^{T-1} \mu_{X_{\tau,\tau+1}} \\
\mu_B &= \mathbb{E} (B) = \mathbb{E} \left( \log \left( \frac{K}{D_t} \right) - \sum_{\tau=t}^{T-1} \log \left( \frac{D_{\tau+1}}{D_{\tau}} \right) \right) = \log \left( \frac{K}{D_t} \right) - \sum_{\tau=t}^{T-1} \mu_{D_{\tau+1}} \\
\sigma^2_A &= \text{Var} (A) = \text{Var} \left( \sum_{\tau=t}^{T-1} \log (X_{\tau,\tau+1}) \right) = \sum_{\tau=t}^{T-1} \text{Var} (\log (X_{\tau,\tau+1})) = \sum_{\tau=t}^{T-1} \sigma^2_{X_{\tau,\tau+1}} \\
\sigma^2_B &= \text{Var} (B) = \text{Var} \left( \log \left( \frac{K}{D_t} \right) - \sum_{\tau=t}^{T-1} \log \left( \frac{D_{\tau+1}}{D_{\tau}} \right) \right) = \sum_{\tau=t}^{T-1} \sigma^2_{D_{\tau+1}} \\
\sigma_{AB} &= \rho_{AB} \left( \sigma^2_A \sigma^2_B \right)^{1/2} = \rho_{AB} \left( \sum_{\tau=t}^{T-1} \sigma^2_{X_{\tau,\tau+1}} \right)^{1/2} \left( \sum_{\tau=t}^{T-1} \sigma^2_{D_{\tau+1}} \right)^{1/2}
\end{align*}
\]

### A.2 List of symbols Part II

- \(h_t\) hazard-rate process
- \(L_t\) fractional expected loss if default occurs at some time \(t\)
- \(r\) risk free interest rate
- \(R_t\) intensity exogenous rate process \(R_t = r + h_t L_t\)
$K$ payoff

$Q$ risk neutral probability measure given the information $\mathcal{I}_t$

$S_t$ asset value

$T$ terminal fixed date

$P_t$ put option price

$\mu$ constant

$\sigma$ constant

$W_t$ Brownian Motion

$m_{t,T}$ stochastic discount factor

$S_t$ payment capacity

$k \equiv \frac{K}{S_t}$ ratio between the contracted debt amount $K$ and the initial payment capacity $S_t$

$U^T = (U_\tau)_{1 \leq \tau \leq T}$ path of economic state variables

$A \equiv \sum_{\tau = t}^{T-1} \log (m_{\tau,\tau+1})$

$B \equiv \log k - \sum_{\tau = t}^{T-1} \log \left( \frac{S_{\tau+1}}{S_{\tau}} \right)$

$p_{i,j} = \Pr [U_{t+1} = j \mid U_t = i]$, transition probabilities

$p_{i,j} = 1 - p_{i,i}$ for $i \neq j$

$\pi_i = \frac{1-p_{ii}}{2-p_{ii}-p_{jj}}$, $i,j = 1,2$.

$(\varepsilon_{1,t}, \varepsilon_{2,t})$ follows a bivariate standard normal distribution with correlation coefficient $\rho_{m,S}$

$\Phi(\cdot)$ cumulative Normal distribution

$\mu_A = \sum_{\tau = t}^{T-1} \mu_{m_{\tau,\tau+1}}$
\[
\mu_B = \log \left( \frac{K}{S_t} \right) - \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} \\
\sigma_A^2 = \sum_{\tau=t}^{T-1} \sigma_{m_{\tau+1}}^2 \\
\sigma_B^2 = \sum_{\tau=t}^{T-1} \sigma_{S_{\tau+1}}^2 \\
\sigma_{AB} = \rho_{AB} \left( \sum_{\tau=t}^{T-1} \sigma_{m_{\tau+1}}^2 \right)^{1/2} \left( \sum_{\tau=t}^{T-1} \sigma_{S_{\tau+1}}^2 \right)^{1/2} \\
d_1 = \frac{\mu_B + \sigma_{AB}}{\sigma_B} \\
d_2 = d_1 - \sigma_B 
\]

- \( m_t^r \) short rate-inflation ratio process

- \( m_t^e \) export-import ratio process
Appendix B

Appendix Part III

B.1 Switching Black and Scholes formula

Proof. Conditioning on $F_C^t$, the price of a European call option contract is derived by the expected discounted value of its payoff under the risk neutral probability measure $Q$, which can be written as

$$\begin{align*}
C(0, T, S(0), K, N) &= \mathbb{E}_Q^C \left[ e^{-\bar{r}T} [S(T) - K]^+ \right] \\
&= \int_{-\infty}^{\infty} e^{-\bar{r}T} [S(T) - K]^+ dQ \\
&= \int_{-\infty}^{\infty} e^{-\bar{r}T} \left[ S(0) e^{Z(T)} - K \right]^+ dQ \\
&= S(0) \int_{\ln(K/S(0))}^{\infty} e^{-\bar{r}T} e^{Z(T)} dQ - K \int_{\ln(K/S(0))}^{\infty} e^{-\bar{r}T} dQ
\end{align*}$$

By equation 13.6

$$\begin{align*}
0 &= S(0) \int_{\ln(K/S(0))}^{\infty} e^{-\bar{r}T} e^{Z(T)} \frac{1}{\sqrt{2\pi \sigma^2(T)}} e^{-\frac{1}{2\sigma^2(T)}[Z(T)-(\bar{r}T)(-T_{\sigma^2(T)})]^2} dZ(T)
\end{align*}$$
\[-K \int_{\ln(K/S(0))}^{\infty} e^{-\bar{T}_r(T)} \frac{1}{\sqrt{2\pi T_{\sigma^2}(T)}} e^{-\frac{1}{2\sigma^2} \left[Z(T)-(\bar{T}_r(T)-\frac{1}{2}T_{\sigma^2}(T))\right]^2} dZ(T)\]

\[= S(0) \text{ Integral } 1 - K \text{ Integral } 2\]

To solve the integrals, consider the following change of variables:

Let \(X(T) = \frac{Z(T)-(\bar{T}_r(T)-\frac{1}{2}T_{\sigma^2}(T))}{\sqrt{T_{\sigma^2}(T)}}\), thus \(dX(T) = \frac{1}{\sqrt{T_{\sigma^2}(T)}} dZ(T)\).

The lower limit of the integral, \(Z(T) = \ln(K/S(0))\) becomes

\[X(T) = \frac{\ln(K/S(0))-(\bar{T}_r(T) - \frac{1}{2}T_{\sigma^2}(T))}{\sqrt{T_{\sigma^2}(T)}} \equiv b.\]

Therefore, Integral 1 becomes

\[
= \int_{b}^{\infty} e^{\frac{1}{2}\frac{X(T)^2}{T_{\sigma^2}(T)}} e^{-\frac{1}{2}X(T)^2} \frac{1}{\sqrt{2\pi T_{\sigma^2}(T)}} dX(T)
= \int_{b}^{\infty} e^{-\frac{1}{2}\left[T_{\sigma^2}(T)-2X(T)\sqrt{T_{\sigma^2}(T)}+X(T)^2\right]} dX(T)
= \int_{b}^{\infty} e^{\frac{1}{2}\left[X(T)-\sqrt{T_{\sigma^2}(T)}\right]^2} dX(T)
\]

Once more, define a change of variable \(h(T) \equiv X(T) - \sqrt{T_{\sigma^2}(T)}\), then \(dh(T) = dX(T)\) and the lower limit of the integral \(X(T) = b\) becomes \(h(T) = b - \sqrt{T_{\sigma^2}(T)} \equiv c.\)

Thus,

\[
\text{Integral } 1 = \int_{c}^{\infty} e^{-\frac{1}{2}h(T)^2} dh(T) = \int_{-\infty}^{-c} e^{-\frac{1}{2}h(T)^2} dh(T) = \mathcal{N}(d_1)
\]

where \(d_1 = -c = \sqrt{T_{\sigma^2}(T)} - b\) and \(\mathcal{N}\) is the Gaussian cumulative distribution function.

While Integral 2 becomes

\[
= \int_{\ln(K/S(0))}^{\infty} e^{-\bar{T}_r(T)} \frac{1}{\sqrt{2\pi T_{\sigma^2}(T)}} e^{-\frac{1}{2\sigma^2} \left[Z(T)-(\bar{T}_r(T)-\frac{1}{2}T_{\sigma^2}(T))\right]^2} dZ(T)
\]
\[ = e^{-T_r(T)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} X(T)^2} dZ(T) = e^{-T_r(T)} \int_{-b}^{b} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} X(T)^2} dZ(T) \]

\[ = e^{-T_r(T)} N(d_2) \]

where \( d_2 = -b = d_1 - \sqrt{T_{\sigma^2}(T)}. \)

Then

\[ C(0, T, S, K, N) = S(0) N(d_1) - Ke^{-T_r(T)} N(d_2) \]

where

\[ d_1 = \frac{\ln(S(0)/K) + T_r(T) + \frac{1}{2} T_{\sigma^2}(T)}{\sqrt{T_{\sigma^2}(T)}} \]

\[ d_2 = \frac{\ln(S(0)/K) + T_r(T) - \frac{1}{2} T_{\sigma^2}(T)}{\sqrt{T_{\sigma^2}(T)}} \]

\[ T_r(T) = \sum_{i=1}^{n} r_i T_i(T); \ T_{\sigma^2}(T) = \sum_{i=1}^{n} \sigma_i^2 T_i(T) \]

\[ \blacksquare \]

**B.2 Pricing on incomplete markets**

Assume that the dynamics of the underlying process \( S(t) \), under the risk-neutral probability measure \( Q \), is given by the stochastic differential equation

\[ dS(t) = r(C(t)) S(t) dt + \sigma(C(t)) S(t) dW(t) \]

where \( W(t) \) is a standard Brownian motion on some filtered probability space \( (\Omega, \{F^W_t\}, Q) \).

The continuous-time Markov process \( \{C(t); t \geq 0\} \) admits the following semi-martingale representation

\[ dC(t) = AC(t) dt + dM(t) \]
where $A(t) = [a_{ij}(t)]$ is the generator of the process $\{C(t), t \geq 0\}$ and $M(t)$ is a $\{\mathcal{F}_t^C\}$-martingale. (See Elliott, Aggoun and Moore (1995) and Spreij (1998)).

Consider a standard European option $V \equiv V(t, S(t), C(t))$. Assume that $W(t)$ and $M(t)$ are two independent processes and denote by $\mathcal{F}_t = \{\mathcal{F}_t^C \vee \mathcal{F}_t^W\}$ the complete filtration generated by $\{C(t), t \geq 0\}$ and $\{W(t), t \geq 0\}$.

Applying Itô’s formula, the following stochastic differential equation is obtained

$$dV = \left( \frac{\partial V}{\partial t} + r(C(t)) S(t) \frac{\partial V}{\partial S} + A C(t) \frac{\partial V}{\partial C} + \frac{1}{2} \sigma^2(C(t)) S^2(t) \frac{\partial^2 V}{\partial S^2} \right) dt$$

$$+ \sigma(C(t)) S(t) \frac{\partial V}{\partial C} dM(t) + \frac{1}{2} \sigma^2(C(t)) S^2(t) \frac{\partial^2 V}{\partial C^2} d\langle M(t) \rangle$$

which becomes

$$dV = \left( \frac{\partial V}{\partial t} + r(C(t)) S(t) \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2(C(t)) S^2(t) \frac{\partial^2 V}{\partial S^2} \right) dt$$

$$+ \sigma(C(t)) S(t) \frac{\partial V}{\partial C} dW(t)$$

$$+ \left( A C(t) \frac{\partial V}{\partial C} + \frac{1}{2} \frac{\partial^2 V}{\partial C^2} \right) dt + \frac{\partial V}{\partial C} dM(t).$$

This equation contains two sources of randomness: the Brownian motion $W(t)$ and the martingale process $M(t)$. The additional risk introduced by the switching on the regime can not be hedged by buying or selling risky assets and risk-free bonds only. Thus, the market is incomplete and there are infinitely many equivalent risk neutral measures.

### B.3 Some definitions and results of Brownian excursions

See Chesney, Jeanblanc and Yor (1997).
Definition 21 **Excursion intervals.** Let $\{ Z ( t ) ; t \geq 0 \}$ be a standard Brownian motion.

For each $t$ define

\[ g_t \equiv \sup \{ s \leq t \mid Z ( s ) = 0 \} \]

\[ d_t \equiv \inf \{ s \geq t \mid Z ( s ) = 0 \} \]

then, almost sure $g_t < t < d_t$ and $(g_t, d_t)$ is the interval of the Brownian excursion which straddles time $t$. For $u$ in this interval, the sign remains unchanged.

Lemma 22 The law of the pair $(g_t, Z ( t ))$ is

\[ 1_{s \leq t} \frac{|x|}{2 \pi \sqrt{s(t-s)^3}} e^{-\frac{x^2}{2(s-t)}} dsdx. \]

Definition 23 **The Brownian meander.** Recall that $g_t$ denotes the left extremity of the Brownian excursion which straddles time $t$. The Brownian meander is defined as the process

\[ m_u^{(t)} = \frac{1}{\sqrt{t-g_t}} |Z ( g_t + u (1-g_t))| ; u \leq 1 \]

in particular the law of $m_u^{(t)}$ does not depend on $t$.

Definition 24 **The Azéma martingale** is the process defined by

\[ \mu_t = \text{sign} (Z ( t )) \sqrt{t-g_t}. \]

Lemma 25 If $\{ Z ( t ) ; t \geq 0 \}$ is a Brownian motion starting from zero, the Laplace transform of the first time a negative Brownian excursion lasts more than $D$ denoted by $\tau^0 \equiv \inf \{ t > 0 \mid (t-g_t) \geq D; Z ( t ) \leq 0 \}$ is

\[ \mathbb{E}^Q \left[ e^{-\theta \tau^0 (Z(t))} \right] = \frac{1}{\Psi \left( \sqrt{2\theta D} \right)} \]
where
\[
\Psi (x) = \int_0^\infty ue^{ux-\frac{1}{2}u^2}du = 1 + x\sqrt{2\pi}e^{\frac{1}{2}x^2}N(x)
\]

\(N(x)\) is the Gaussian cumulative distribution function.

### B.4 Examples of pricing on incomplete markets

#### B.4.1 Mean-variance criteria. Di Masi et al. (1994).

Di Masi et al. derived the PDE that governs the switching underlying process, then they follow the approach based on the idea of hedging under a mean-variance criterion.

#### B.4.2 PDE approach. Boyle and Draviam (2007)

They assume that the extra uncertainty is not priced, then the following partial differential equation is derived.

\[
\frac{\partial V}{\partial t} + r(C(t))S(t) \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(C(t))S^2(t) \frac{\partial^2 V}{\partial S^2} + \langle V, AC(t) \rangle = r(C(t))V
\]

#### B.4.3 Complete the market. Guo (2001)

Guo (2001) proposed to complete the market, by introducing a contract that pays one unit when the state switches from one state to another.

#### B.4.4 Esscher transform

Use the Esscher transform to price the option as in Gerber and Shiu (1994). Measure with minimal entropy.
B.4.5 Stochastic discount factors

B.5 List of symbols Part III

Continuous-time Markov chain

$(\Omega, \mathcal{F}, P)$ probability space

$\{C(t), t \geq 0\}$ continuous-time Markov chain process

$\mathcal{E} = \{e_1, ..., e_N\}$ discrete state space of $N$ elements

$e_i = [0, ..., 0, 1, 0, ... 0]^T$, $N \times 1$ vector

$(\cdot, \cdot)$ inner product of vectors

$\mathcal{F}_t^C = \sigma(\{C(t)\})$ be the complete filtration generated by the process $\{C(t), t \geq 0\}$.

$p_{ij}(t)$ transition probability from state $i$ to state $j$ at a time $t$ later

$P(t) = [p_{ij}(t)]$ transition probability $N \times N$ matrix

$I$ identity matrix

$A(t) = [a_{ij}(t)]$, is a $N \times N$ generator or Q-matrix of the process

$p(0) = (p_1(0), ..., p_i(0), ..., p_N(0))^T$ vector of initial probabilities

$M(t)$ an $\{\mathcal{F}_t\}$ martingale.

Occupational time for Markov chains

$1_{[C(t) = e_i]}$ denotes the indicator function having value of one if the chain is in state $i$ at time $t$ and zero otherwise.

$T_i(t) = \int_0^t \langle e_i, C(s) \rangle \, ds$ total time the continuous-time Markov chain $C(t)$ spends in state $i$ over the interval $[0, t]$

$\bar{T}(t) = \sum_{i=1}^n d_i T_i(t)$ weighted sum of the times spent in the states $i = 1, ..., n; n \leq N$, 
during the time interval $[0, t]$

$d_i$ positive weights

$G_{\bar{T}(t)}(\theta)$ moment generating function of $\bar{T}(t)$

$G_{T_i}(\theta)$ moment generating function of $T_i(t)$

$D_1 = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ a $N \times N$ matrix

$D = (d_i)$ a $n \times n$ matrix, $i = 1, ..., n; n \leq N$

$D_r$ is a $N \times N$ matrix with elements in the diagonal $(r_i)$, been the return rates on the riskless asset or bond, for $i = 1, ..., N$.

$D_\mu$ is a $N \times N$ matrix with elements in the diagonal $(\mu_i)$, been the return rates on the risky asset or stock for $i = 1, ..., N$.

$1 = [1, ..., 1]'$ a $N \times 1$ vector of ones

**Dynamics**

$B^0(t)$ riskless asset or bond price

$\{S(t); t \geq 0\}$ underlying stochastic process representing the stock price or risky asset at time $t$

$W(t)$ standard Brownian motion

$\mathcal{F}_t^S = \sigma(\{S(t)\})$ filtration generated by the process $\{S(t), t \geq 0\}$

$\mathcal{F}_t = \{\mathcal{F}_t^C \vee \mathcal{F}_t^S\}$ complete filtration generated by the process $\{C(t), t \geq 0\}$ and $\{S(t), t \geq 0\}$

$Q$ risk neutral probability measure

$Z(t) \equiv \ln(S(t)/S(0))$

$\mu = (\mu_1, ..., \mu_N)'$ return rate of the risky asset; $\mu_i$ constant
\( \sigma = (\sigma_1, ..., \sigma_N)' \) volatility of the risky asset; \( \sigma_i \geq 0 \) constant

\( r = (r_1, ..., r_N)' \) instantaneous risk-free rate of the bond; \( r_i > 0 \)

\( \mu (C(t)) = \langle \mu, C(t) \rangle \)

\( \sigma (C(t)) = \langle \sigma, C(t) \rangle \)

\( r (C(t)) = \langle r, C(t) \rangle \)

\( \bar{T}_r(t) = \sum_{i=1}^{N} r_i T_i(t) = \int_{0}^{t} r(C(s)) \, ds \) weighted sum of time spent in states \( i = 1, ..., N \) with weight \( d_i = r_i \)

\( \bar{T}_\mu(t) = \sum_{i=1}^{N} \mu_i T_i(t) = \int_{0}^{t} \mu(C(s)) \, ds \) weighted sum of time spent in states \( i = 1, ..., N \) with weight \( d_i = \mu_i \)

\( \bar{T}_{\sigma^2}(t) = \sum_{i=1}^{N} \sigma_i^2 T_i(t) = \int_{0}^{t} \sigma^2(C(s)) \, ds \) weighted sum of time spent in states \( i = 1, ..., N \) with weight \( d_i = \sigma_i \)

\( \bar{T}_{\sigma}(W(t)) = \sum_{i=1}^{N} \sigma_i W(T_i) = \int_{0}^{t} \sigma(C(s)) \, dW(s) \)

\( G_{Z(t)}(\theta) \) moment generating function of \( Z(t) \)

**Parisian activation**

\( T \) maturity time of the option

\( r \) risk-free rate

\( L \) barrier level

\( D \) activation or deactivation window (grace period)

\( - \) indicates the process is below the level \( L \) (Down)

\( + \) indicates the process is above the level \( L \) (Up)

\( in \) indicates the option is ”in the money”

\( out \) indicates the option is ”out of the money”
\( \phi () \) \hspace{1cm} \text{payoff function} \\

\( PP_{in}^- \) \hspace{1cm} \text{Parisian Put option Down and In} \\

\( PP_{out}^- \) \hspace{1cm} \text{Parisian Put option Down and Out} \\

\( PP_{in}^+ \) \hspace{1cm} \text{Parisian Put option Up and In} \\

\( PP_{out}^+ \) \hspace{1cm} \text{Parisian Put option Up and Out} \\

\( PC_{in}^- \) \hspace{1cm} \text{Parisian Call option Down and In} \\

\( PC_{out}^- \) \hspace{1cm} \text{Parisian Call option Down and Out} \\

\( PC_{in}^+ \) \hspace{1cm} \text{Parisian Call option Up and In} \\

\( PC_{out}^+ \) \hspace{1cm} \text{Parisian Call option Up and Out} \\

**Occupational time for Parisian options**

\( T \) \text{ maturity time of the option} \\

\( T^+ \) \text{total time spent above the barrier} \( L \) \\

\( T^- \) \text{total time spent below the barrier} \( L \) \\

\( D \) a fixed grace period, \( D \) constant \\

\( g_t^L(X) \) \text{last time before} \( t \) \text{the process} \( X \) \text{hits the barrier} \( L \), \text{i.e.} \( g_t^L(X) \equiv \sup\{s \leq t \mid X_s = L\} \) \\

\( G_{D,L}^-(X) \) \text{first time} \text{the excursion time} \( (t-g_t^L(X)) \) \text{exceeds} \( D \); \text{i.e.} \( G_{D,L}^-(X) \equiv \inf\{t > 0 \mid t-g_t^L(X) \geq D\} \) \\

\( 1_{G_{D,L}^-(X)<T} \) \text{activation of a Parisian call and down option} \( PC_{in}^- \) \text{before maturity} \( T \), \text{i.e.} \\

\[
1_{G_{D,L}^-(X)<T} = \begin{cases} 
1 & \text{if } G_{D,L}^-(X) < T \\
0 & \text{otherwise}
\end{cases}
\]
Appendix C

Appendix Part IV

C.1 Valuation

From Section 2, consider the liabilities and equity’s value equations
\[ V_L (0, T, S, L, N) = G - PO + BO + RB \]
\[ V_E (0, T, S, L, N) = CO - BO \]

We develop each term:

\[ G = \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} L(T) \mathbf{1}_{\tau \geq T} \right] \]
\[ = e^{-\bar{T}_r(T)} L(T) \left[ 1 - \mathbb{Q}(\tau < T) \right] \]
\[ = e^{-\bar{T}_r(T)} L(T) \left[ 1 - \mathbb{Q}_1 \right] \]

\[ PO = \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \left[ L(T) - S(T) \right]^+ \mathbf{1}_{\tau \geq T} \right] \]
\[ = \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \left[ L(T) - S(T) \right]^+ \right] - \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \left[ L(T) - S(T) \right]^+ \mathbf{1}_{\tau < T} \right] \]
\[ = \text{SWPO} [L(T)] - e^{-\bar{T}_r(T)} L(T) \mathbb{Q}[S(T) < L(T); \tau < T] \]
\[
+ \mathbb{E}^Q \left[ e^{-T_r(T)} S(T) \mathbf{1}_{S(T) < L(T)} \mathbf{1}_{\tau < T} \right] \\
= SWPQ \left[ L(T) \right] - e^{-T_r(T)} L(T) Q_2 + \mathbb{E}_3^Q
\]

\[
BO = \mathbb{E}^Q \left[ \alpha \delta e^{-T_r(T)} \left( S(T) - \frac{L(T)}{\alpha} \right)^+ \mathbf{1}_{\tau \geq T} \right] \\
= \mathbb{E}^Q \left[ \alpha \delta e^{-T_r(T)} \left( S(T) - \frac{L(T)}{\alpha} \right)^+ \right] - \mathbb{E}^Q \left[ \alpha \delta e^{-T_r(T)} \left( S(T) - \frac{L(T)}{\alpha} \right)^+ \mathbf{1}_{\tau < T} \right] \\
= \alpha \delta SWCO \left[ \frac{L(T)}{\alpha} \right] - \alpha \delta \mathbb{E}_4^Q + \delta e^{-T_r(T)} L(T) Q_5
\]

\[
RB = \min \{ \lambda, 1 \} L(0) \mathbb{E}^Q \left[ e^{-T_r(T) + \lambda \tau} \mathbf{1}_{\tau < T} \right] = \min \{ \lambda, 1 \} L(0) \mathbb{E}_6^Q
\]

\[
CO = \mathbb{E}^Q \left[ e^{-T_r(T)} \left( S(T) - L(T) \right)^+ \mathbf{1}_{\tau \geq T} \right] \\
= \mathbb{E}^Q \left[ e^{-T_r(T)} \left( S(T) - L(T) \right)^+ \right] - \mathbb{E}^Q \left[ e^{-T_r(T)} \left( S(T) - L(T) \right)^+ \mathbf{1}_{\tau < T} \right] \\
= SWCO \left[ L(T) \right] - \mathbb{E}^Q \left[ e^{-T_r(T)} S(T) \mathbf{1}_{S(T) > L(T)} \mathbf{1}_{\tau < T} \right] \\
+ e^{-T_r(T)} L(T) Q \left[ S(T) > L(T) ; \tau < T \right] \\
= SWCO \left[ L(T) \right] - \mathbb{E}^Q_8 + e^{-T_r(T)} L(T) Q_8
\]

where

\[
Q_1 \equiv Q \left( \tau < T \right) \quad (a1)
\]

\[
Q_2 \equiv Q \left[ S(T) < L(T) ; \tau < T \right] \quad (a2)
\]

\[
\mathbb{E}^Q_3 \equiv \mathbb{E}^Q \left[ e^{-T_r(T)} S(T) \mathbf{1}_{S(T) < L(T)} \mathbf{1}_{\tau < T} \right] \quad (a3)
\]
\[ E^Q_4 \equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > \frac{L(T)}{\alpha}} \mathbf{1}_{\tau < T} \right] \] (a4)

\[ Q_5 \equiv \mathbb{Q} \left[ S(T) > \frac{L(T)}{\alpha} ; \tau < T \right] \] (a5)

\[ E^Q_6 \equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(\tau) + r_g \tau} \mathbf{1}_{\tau < T} \right] \] (a6)

\[ E^Q_7 \equiv \mathbb{E}^Q \left[ e^{-T_r(T) S(T)} \mathbf{1}_{S(T) > L(T)} \mathbf{1}_{\tau < T} \right] \] (a7)

\[ Q_8 \equiv \mathbb{Q} \left[ S(T) > L(T) ; \tau < T \right] \] (a8)

\[ \text{SWCO}[K] \equiv \text{switching call option at } T; \text{ strike } K \]

\[ \text{SWPO}[K] \equiv \text{switching put option at } T; \text{ strike } K \]

\[ e^{-\bar{T}_r(T)} L(T) = L(0) e^{-T_g T} \text{ where} \]

\[ \bar{T}_g \equiv [r_1 - r_g, \ldots, r_N - r_g]' \]

It follows that

\[ V_L(0, T, S, L, N) = e^{-\bar{T}_r(T)} L(T) \left[ 1 - Q_1 + Q_2 + \delta Q_5 \right] - E^Q_3 - \alpha \delta E^Q_4 + \min \left\{ \lambda, 1 \right\} L(0) E^Q_6 \]

\[ + \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} \right] - \text{SWPO} [L(T)] \]

\[ V_E(0, T, S, L, N) = e^{-\bar{T}_r(T)} L(T) \left[ Q_8 - \delta Q_5 \right] + \alpha \delta E^Q_4 - E^Q_7 \]

\[ + \text{SWCO} [L(T)] - \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} \right] \]
C.2 Closed-form formula for regime switching European options

The formulas for the call and put options with underlying switching process, strike price \(K\) and maturity date \(T\) are respectively

\[
\begin{align*}
  \text{SWCO}[K] &= S(0)N(d_1) - Ke^{-\tilde{T}_r(T)}N(d_2) \\
  \text{SWPO}[K] &= Ke^{-\tilde{T}_r(T)}N(-d_2) - S(0)N(-d_1)
\end{align*}
\]

where

\[
\begin{align*}
  d_1 &= \frac{\log(S(0)/K) + \tilde{T}_r(T) + \frac{1}{2}\tilde{T}_{\sigma^2}(T)}{\sqrt{\tilde{T}_{\sigma^2}(T)}} \\
  d_2 &= \frac{\log(S(0)/K) + \tilde{T}_r(T) - \frac{1}{2}\tilde{T}_{\sigma^2}(T)}{\sqrt{\tilde{T}_{\sigma^2}(T)}} = d_1 - \sqrt{\tilde{T}_{\sigma^2}(T)} \\
  \tilde{T}_r(T) &= \sum_{i=1}^{N} r_i T_i(T); \quad \tilde{T}_{\sigma^2}(T) = \sum_{i=1}^{N} \sigma_i^2 T_i(T);
\end{align*}
\]

\(N\) is the Gaussian cumulative distribution function and \(T_i(T)\) denotes the amount of time the continuous-time Markov chain spends on state \(i\) over the time period interval \([0, T]\), \(i = 1, ..., N\). See Monter (2008a).

To solve equations (a1) to (a8), the law the liquidation time \(\tau\) is needed.

The random variable \(\tau\) is defined as the first time the underlying regime switching process \(S(t)\) hits the barrier \(B(t)\) during the life of the option:

\[
\tau = \inf \{t \in [0, T] \mid S(t) \leq B(t)\} \quad ; \quad B(t) \equiv \lambda L(0) e^{\sigma t}
\]
C.3 Numerical procedure

Since a closed form solution is difficult to obtain, numerical scheme is proposed for a piecewise evaluation:

1. Discretize the time interval $[0, T]$ on $m$ intervals of size $\Delta = \frac{T}{m}$. Note that $m$ must be large enough such that each sub-interval of time $\Delta_j$ for $j = 1, ..., m$ allows at most one switching of state. Define $t_j = j\Delta$, so $t_1 = \Delta, t_2 = 2\Delta, ..., t_m = m\Delta = T$.

2. Set the number of simulations be equal to $l$.
   
   (a) Given the probability matrix $A(t)$, simulate a single path of the $\{C_j(t); 0 \leq t \leq T; j = 1, ..., m\}$ process taking values on the $i = 1, ..., N$ possible states.
   
   (b) Compute the occupational time $T_i(T)$ for each state $i = 1, ..., N - 1$.
   
   (c) Compute $\bar{T}_r(T) = \frac{1}{N} \sum_{i=1}^{N} r_i T_i(T)$ and $\bar{T}_{\sigma^2}(T) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 T_i(T)$.
   
   (d) Applied the weighted Black and Scholes formula.

3. Repeat step 3 for $l$ times and get the average of the weighted Black and Scholes formula.

Alternatively, for two states regime switching model:

1. Compute the expected value of each one, from the moment generating function of $\bar{T}_r(T)$ and $\bar{T}_{\sigma^2}(T)$.

2. Plug the results into the weighted Black and Scholes formula.
C.4 Geometric Brownian motion parameter estimation

The firm’s assets’ value is specified by the equation

$$dS(t) = \mu S(t) \, dt + \sigma S(t) \, dW(t)$$

(C.1)

where $W(t)$ is a standard Brownian motion. Itô’s formula implies

$$\log \left( \frac{S(t)}{S(t-1)} \right) = \mu - \frac{\sigma^2}{2} + \sigma (W(t) - W(t-1))$$

Let $R(t) = \log \left( \frac{S(t)}{S(t-1)} \right)$ denotes the log return over one period of time, then

$$R(t) = \mu - \frac{\sigma^2}{2} + \sigma (\varepsilon(t))$$

(C.2)

where $\varepsilon(t)$ follows a standard Gaussian distribution.

The mean and variance maximum likelihood estimators of the log increments are, respectively

$$\hat{m}(T) = \frac{1}{T} \sum_{t=1}^{T} R(t) \quad \hat{s}^2(T) = \frac{1}{T} \sum_{t=1}^{T} [R(t) - \hat{m}(T)]^2$$

therefore, the maximum likelihood estimators of the volatility and drift parameters are given by

$$\hat{\sigma}^2(T) = \hat{s}^2(T), \quad \hat{\mu}(T) = \hat{m}(T) + \frac{\hat{s}^2(T)}{2}$$

When data is sampled at a time interval $h$, we get

$$\hat{\sigma}^2(T) = \frac{1}{h} \hat{s}^2(h,T), \quad \hat{\mu}(h,T) = \frac{\hat{m}(h,T)}{h} + \frac{\hat{s}^2(h,T)}{2h}$$

(C.3)

C.5 Daily data analysis

Additional analysis is performed with stock prices time series going from January 2nd, 2003 to April 21st, 2008. More than 1300 daily observations.
C.6 Liquidation allowing a grace period: Parisian options

Although it has been already a significant progress towards modelling default of the firm insurance prior to maturity by introducing a barrier, the work of Chen and Suchanecki (2007) goes further by considering not only the first time the firm’s assets’ value hits the barrier, but by allowing a grace period. Once the firm’s assets’ value has hit the barrier, a clock starts counting the time the firm’s assets’ value spent below this barrier, then, the first time it exceeds a given grace period, liquidation arrives. If the firm recovers before the grace period finish, then the clock is reset to zero. In this setting, similar than credit risk models (See François and Morellec (2004)), default is distinguished from liquidation due to the grace period. Liquidation time is defined as \( \tau = \inf \{ t \mid (t - g) \geq d; S(t) < B(t) \} \) for \( 0 < t < T \), where \( d \) is a fixed grace period, and \( g = \sup \{ s \leq t \mid S(s) = B(s) \} \) denotes the last time the process hit the barrier.

Hence, the evaluation of the equity holder and liability holder may be written as a combination of Parisian options, created originally by Chesney, Jeanblanc and Yor (1997).